AUTOMAATJUHTIMINE JA SÜSTEEMIANALÜÜS

Eduard Petlenkov, 2019

Diskreetaja süsteemid

mudelid ja analüüs

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Pidevaja süsteemi mudelid

Ülekandemudelid:
[sisend–väljund mudelid]
- diferentsiaalvõrrand / dif.võrrandite süsteem
- ülekandefunktsioon / ülekandemaatriks
- hüppekaja / hüppekajade maatriks
- impulsskaja / impulsskajade maatriks

Olekumudel:
[sisend-olek-väljund mudel]
- olekuvõrrand
- väljundvõrrand

Poolused [ülekande-funktsiooni nimetaja juured] ↔ **Omaväärtused** [oleku-võrrandi A maatriksi oma-väärtused]

Poolused / omaväärtused määravad süsteemi käitumise.

Pidevaja süsteemi olekumudel

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), x(0) \end{cases} \quad \text{kus} \quad A - nxn; \; B - nxr; \; C - mxn;$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \; u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}; \; y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}.$$

$$\begin{cases} x(t) = \underbrace{e^{At}x(0)}_{\substack{vabaliiku- \\ \text{mine} \leftarrow x(0)}} + \int_0^t \underbrace{e^{A(t-\tau)}Bu(\tau)d\tau}_{\substack{sundliiku- \\ \text{mine} \leftarrow u(t)}} \\ \mathrm{y(t) = Cx(t)} \end{cases}$$

Ekvivalentne diskreetne süsteem (1)

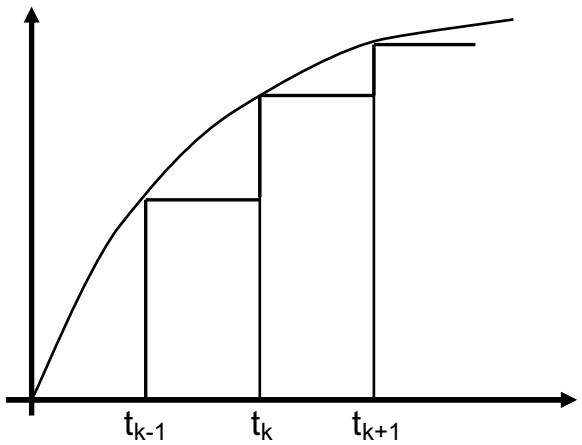
Antud on pidevaja süsteemi olekumudel

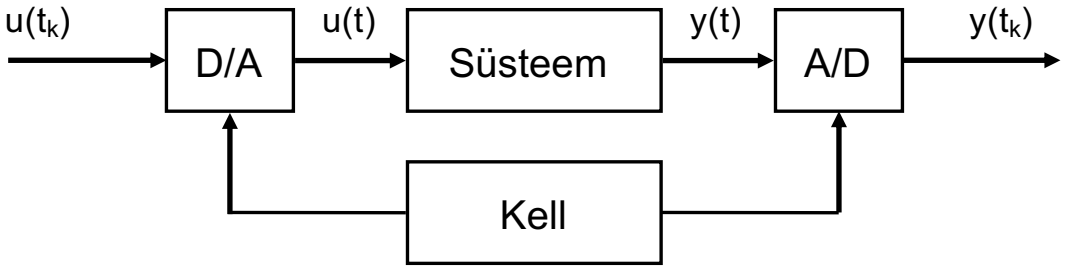
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t),\ x(0) \end{cases} \quad \text{kus} \quad A - nxn;\ \ B - nxr;\ \ C - mxn.$$

$$u(t) \to u(t_k)$$

$$x(t) \to x(t_k)$$

$$y(t) \to y(t_k)$$





Ekvivalentne diskreetne süsteem (2)

Lähtume pidevaja olekumudeli lahendist

$$x(t) = e^{At}x(0) + \int_0^t e^{A\tau}Bu(t-\tau)d\tau$$

- $t_k \rightarrow t$ (üleminek)

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

- $t \rightarrow t_{k+1}$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau =$$

$$= e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}d\tau\, Bu(t_k)$$

NB! $t_k \rightarrow t_{k+1}$ $u(\tau)=u(t_k)$

Ekvivalentne diskreetne süsteem (3)

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k) \cdot x(t_k) + \Gamma(t_{k+1}, t_k) u(t_k),$$

kus $\Phi(t_{k+1}, t_k) = e^{A(t_{k+1} - t_k)}$,

$$\Gamma(t_{k+1}, t_k) = \int_0^{t_{k+1} - t_k} e^{A\tau} d\tau \cdot B.$$

Eeldame $t_{k+1} - t_k = h \quad k = 0,1,2,\cdots$

teisiti $t_k = kh \quad k = 0,1,2,\cdots.$

$$\begin{cases} x(kh + h) = \Phi x(kh) + \Gamma u(kh) \\ y(kh) = Cx(kh), \quad x(0); \end{cases}$$

kus $\Phi = e^{Ah}$,

$$\Gamma = \int_0^h e^{A\tau} d\tau \, B,$$ ekvivalentne diskreetne süsteem.

Ekvivalentne diskreetne süsteem (3)

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k), \;\; x(0) \end{cases}$$

$\underline{h!}$

Probleem – h valik!?

Olekuvõrrandi lahendamine: $k=k_0$ (→ tavaliselt $k_0=0$)

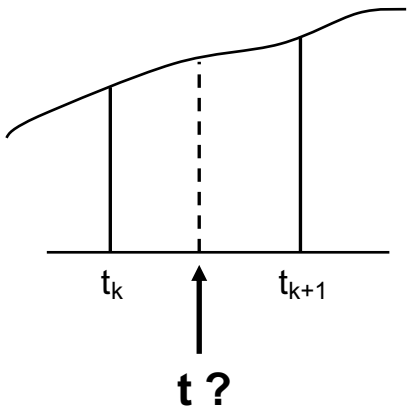
$$x(k_0+1) = \Phi x(k_0) + \Gamma u(k_0)$$

$$x(k_0+2) = \Phi x(k_0+1) + \Gamma u(k_0+1) =$$

$$= \Phi^2 x(k_0) + \Phi\Gamma u(k_0) + \Gamma u(k_0+1)$$

$$x(k) = \Phi^{k-k_0} x(k_0) + \Phi^{k-k_0-1}\Gamma u(k_0) + \cdots \Gamma u(k-1) =$$

$$= \Phi^{k-k_0} x(k_0) + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma u(j)$$



$$x(t) = \Phi(t,t_k)x(t_k) + \Gamma(t,t_k)u(t_k),$$

kus $\quad \Phi(t,t_k) = e^{A(t-t_k)},$

$$\Gamma(t,t_k) = \int_0^{t-t_k} e^{A\tau} d\tau\, B.$$

NB! h valitakse Shannon-Kotelnikovi teoreemi alusel.

Näide No.1 Φ ja Γ arvutamine

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases}$$

kus $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

Leiame Φ ja Γ üldkujul

$$\Phi = e^{Ah} = E + Ah + \frac{A^2h^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\Gamma = \int_0^h e^{A\tau} d\tau\, B = \int_0^h \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

$$\begin{cases} x(kh+h) = \underbrace{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}}_{\Phi} x(kh) + \underbrace{\begin{bmatrix} h^2/2 \\ h \end{bmatrix}}_{\Gamma} u(kh) \\ y(kh) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(kh) \end{cases}$$

Φ ja Γ arvutasime e^{Ah} astmerea alusel.

Meil on antud olekumudel

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k), \quad x(0) \end{cases}$$

Soovime leida ülekandemudeli (sisend-väljund mudeli).

Võtame kasutusele operaatori z

$$y(k) \qquad zy(k) = y(k+1)$$

$$z^{-1}y(k) = y(k-1)$$

$$x(k+1) = zx(k) = \Phi x(k) + \Gamma u(k)$$

$$zx(k) - \Phi x(k) = \Gamma u(k)$$

$$(zE - \Phi)x(k) = \Gamma u(k)$$

$$x(k) = \underbrace{(zE - \Phi)^{-1}\Gamma}_{Hux(z)} u(k)$$

$$y(k) = Cx(k) = \underbrace{C(zE - \Phi)^{-1}\Gamma}_{H(z), Huy(z)} u(k)$$

H(z) – ülekandemaatriks

– m x r.

Eeldame, et m=r=1 → ühemõõtmeline süsteem.

$$H(z) = \frac{B(z)}{A(z)}$$ ülekandefunktsioon

$$= \frac{B(z^{-1})}{A(z^{-1})} = \frac{b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

$$H(z) = \frac{y(k)}{u(k)}$$

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_1 u(k-1) + \cdots + b_n u(k-n),$$

diferentsvõrrand.

Kui u(k)$\cdots$ on antud, siis y(k) $\cdots$on leitavad

$$y(k) = -a_1 y(k-1) - \cdots - a_n y(k-n) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

Näide No.2

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k), \end{cases} \qquad h!$$

kus $\Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \ \Gamma = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

$$H(z) = C(zE - \Phi)^{-1}\Gamma$$

$$zE - \Phi = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z-1 & -h \\ 0 & z-1 \end{bmatrix}$$

Olgu h=1, siis $zE - \Phi = \begin{bmatrix} z-1 & -1 \\ 0 & z-1 \end{bmatrix}, \ (zE-\Phi)^{-1} = \dfrac{1}{(z-1)^2}\begin{bmatrix} z-1 & -1 \\ 0 & z-1 \end{bmatrix}$

$$H(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(z-1)^2}\begin{bmatrix} z-1 & 1 \\ 0 & z-1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \frac{\frac{1}{2}(z+1)}{(z-1)^2} =$$

$$= \frac{0.5z + 0.5}{z^2 - 2z + 1} = \frac{0.5z^{-1} + 0.5z^{-2}}{1 - 2z^{-1} + z^{-2}}$$

$$H(z)=\frac{0.5z^{-1}+0.5z^{-2}}{1-2z^{-1}+z^{-2}} \leftrightarrow \frac{y(k)}{u(k)}$$

$$y(k)-2y(k-1)+y(k-2)=0.5u(k-1)+0.5u(k-2)$$

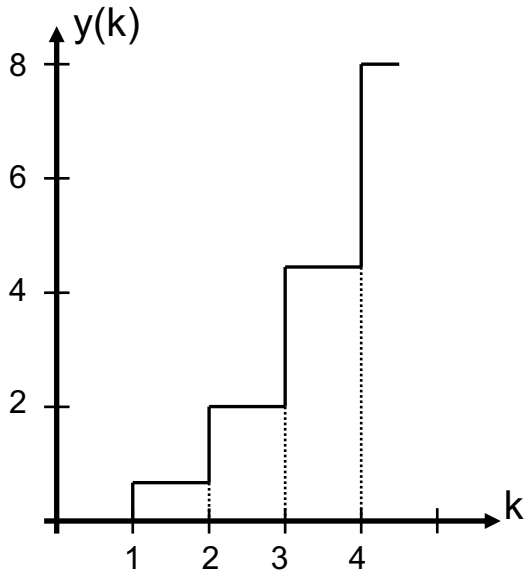
$$y(k)=2y(k-1)-y(k-2)+0.5u(k-1)+0.5u(k-2)$$

$$y(k)=f\left[y(k-1),y(k-2),u(k-1),u(k-2)\right].$$

Süsteemi järk on 2.

u(k)=1, k≥0

k	$y(k-2)$	$y(k-1)$	$u(k-2)$	$u(k-1)$	$y(k)$
0	0	0	0	0	0
1	0	0	0	1	0.5
2	0	0.5	1	1	2
3	0.5	2	1	1	4.5
4	2	4.5	1	1	8



reaalaeg !?

h=1 → k=4 → kh=4

h≠1 → k → kh=t

z-teisendus

$\{x(k);\ k = 0,1,2,\cdots\}$ jada

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

kujutis originaal

1952 – 1958

Jury

Barker

Tsõpkin

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k), \ x(0) \end{cases}$$

$$\sum_{k=0}^{\infty} z^{-k} x(k+1) = z\left[\sum_{k=0}^{\infty} z^{-k} x(k+1) - x(0)\right] =$$

$$= \sum_{k=0}^{\infty} \Phi z^{-k} x(k) + \sum_{k=0}^{\infty} \Gamma z^{-k} u(k)$$

$$\sum_{k=0}^{\infty} x(k) z^{-k} = X(z)$$

$$\sum_{k=0}^{\infty} u(k) z^{-k} = U(z)$$

$$z\left[X(z) - x(0)\right] = \Phi X(z) + \Gamma U(z)$$

$$X(z) = \underbrace{(zE - \Phi)^{-1} z x(0)} + \underbrace{(zE - \Phi)^{-1} \Gamma U(z)}$$

$$H(z) = C(zE - \Phi)^{-1} \Gamma$$

Mõned z-teisendused:

$$x(k)=1,\ \ k\geq 0 \rightarrow X(z)=\frac{z}{z-1},\qquad x(kh)=kh,\ \ k\geq 0 \rightarrow X(z)=\frac{hz}{(z-1)^2}$$

Aeg - **pidev** Aeg - **diskreetne**

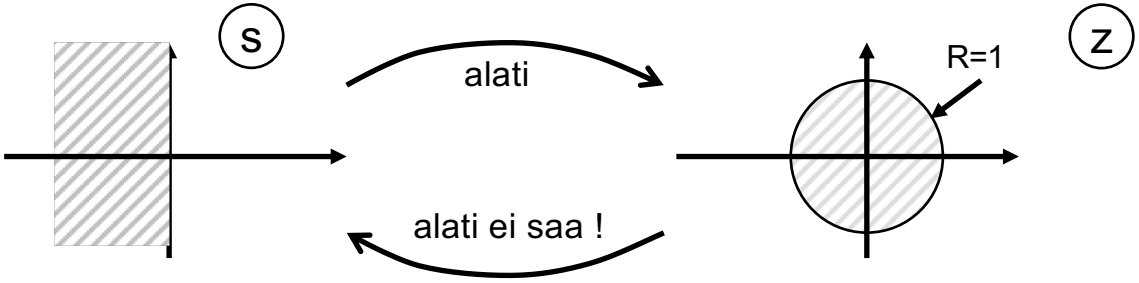
$$x(t) \xrightarrow{h} x(k)$$

$$X(s)=L[x(t)] \qquad X(z)=Z[x(k)]$$

$$\begin{cases} X(s)=\int_0^\infty x(t)e^{-st}dt \\ s=\dot{\tau}+j\omega \end{cases} \qquad \begin{cases} X(z)=\sum_{k=0}^{\infty} x(k)z^{-k} \\ z=\rho+j\vartheta \end{cases}$$

$x(t),\ \ t\geq 0$ $\qquad$ $x(k),\ \ k\geq 0$

$z=e^{sh}$, h-diskreetimissamm



Näide No.3

x(k) arvutamine X(z) alusel.

$$\begin{cases} X(z) = \dfrac{10}{z^2 - 1.5z + 0.5} = \dfrac{10z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}} \\ X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \end{cases}$$

$$\begin{array}{l|l}
10 & z^2 - 1.5z + 0.5 \\ \hline
 & 10z^{-2} + 15z^{-3} + 17.5z^{-4} + \cdots \\
- & 10 - 15z^{-1} + 5z^{-2} \\ \hline
 & 0 + 15z^{-1} - 5z^{-2} \\
- & \quad 15z^{-1} + 22.5z^{-2} + 7.5z^{-3} \\ \hline
 & \quad 0 + 17.5z^{-2} - 7.5z^{-3} \\
- & \qquad 17.5z^{-2} - 26.25z^{-3} + 8.75z^{-4} \\ \hline
 & \qquad 0 + 18.75z^{-3} - 8.75z^{-4}
\end{array}$$

$x(0)=0$

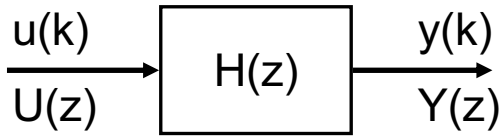
$x(1)=0 \quad \leftarrow x(h)$

$x(2)=10 \quad \leftarrow x(2h)$

$x(3)=15 \quad \leftarrow x(3h)$

$x(4)=17.5 \leftarrow x(4h)$

TAL TECH



Y(z)=H(z)·U(z)

Süsteemifunktsioonid diskreetaja süsteemides (nullised algtingimused):

$$H(s)=\frac{B(s)}{A(s)} \qquad H(z)=\frac{B(z)}{A(z)}=\frac{B(z^{-1})}{A(z^{-1})}$$

Lugeja järk võib olla nimetaja järguga võrdne

Neg.astmete puhul: lugeja ja nimetaja järgud võrdsed

$$h(t)=L^{-1}\left[H(s)\right] \qquad h(k)=z^{-1}\left[H(z)\right]$$

$$g(t)=L^{-1}\left[\frac{H(s)}{s}\right] \qquad g(k)=z^{-1}\left[H(z)\cdot\frac{z}{z-1}\right]$$

H(s)/ H(z) –ülekandefunktsioon, *h(t)/ h(k)* – impulsskaja, *g(t) / g(k)* – hüppekaja.