ISS0031 Modeling and Identification

Lecture 9

Basic concepts

Definition 1. ***Input*** *is something (such as power, energy, or information) that is put into a machine or system .*

Definition 2. ***Output*** *is something that is produced by a machine or system.*

Definition 3. *The* ***state*** *of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.*

Definition 4. *The* ***state variables*** *of a dynamic system are the variables that making up the smallest set of variables that determine the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.*

Definition 5. *The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, …, x_n axis is called a* ***state space****. Any state can be represented by a point in the state space.*

Mathematical models of the linear time-invariant systems

State-space form: A linear time-invariant system (LTI) can be represented by state equations as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

in continuous-time case or by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2}$$

in discrete-time case. Both in (1) and (2) $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is a *state vector*, $u(t) = [u_1(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ is an *input(control) vector*, $y(t) = [y_1(t), \ldots, y_p(t)]^T \in \mathbb{R}^p$ is an *output vector*, A is a *state matrix* with $\dim A = n \times n$, B is a *input matrix* with $\dim B = n \times m$, C is a *output matrix* with $\dim C = p \times n$, D is a *feedthrough matrix* with $\dim D = p \times m$.

System is called single-input single-output (SISO) if $p = m = 1$ or multi-input multi-output (MIMO) if $p, m > 1$.

Input-output form: A linear time-invariant system can be represented by differential input-output equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = b_0 u^{(n)}(t) + \cdots + b_n u(t) \tag{3}$$

in continuous-time case or by difference input-output equation

$$y(t+n) + a_1 y(t+n-1) + \cdots + a_n y(t) = b_0 u(t+n) + \cdots + b_n u(t) \tag{4}$$

in discrete-time case.

Transfer function: A transfer function of a linear continuous-time system is defined as the ratio of the output of a system to the input of a system

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}, \tag{5}$$

where $U(s)$ and $Y(s)$ denote Laplace transform[1] of input and output signals, respectively.

Properties of LTI systems

Controllability and observability represent two major concepts of modern control system theory.

Theorem 1. *The linear* (1) *or* (2) *system is controllable if and only if the controllability matrix has full rank, i.e.*

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n. \tag{6}$$

In other words, controllability condition means that it is possible to choose input signals which allows to steer the states from any initial value to any final value within some finite time window.

Theorem 2. *The linear* (1) *or* (2) *system is observable if and only if the observability matrix has full rank, i.e.*

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{7}$$

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs. In other words, this means that from the system's outputs it is possible to determine the behavior of the entire system. If a system is not observable, this means the current values of some of its states cannot be determined through output sensors.

[1] Z-transform can be used to find a transfer function of a linear discrete-time system.

The stability of a system relates to its response to inputs or disturbances. A system which remains in a constant state unless affected by an external action and which returns to a constant state when the external action is removed can be considered to be stable.

Definition 6. *A system is stable if its impulse response approaches zero as time approaches infinity.*

Definition 7. *A system is stable if every bounded input produces a bounded output.*

Example 1: Consider the following model of a DC motor

$$\begin{aligned} \dot{x}_1 &= -\frac{b}{J}x_1 + \frac{K}{J}x_2 \\ \dot{x}_2 &= -\frac{K}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \\ y &= x_1 \end{aligned} \tag{8}$$

where x_1 is the rotational speed, x_2 is the electric current and u is the voltage. Define parameters as:

- moment of inertia of the rotor $J = 0.01$ kg.m^2/s^2,
- damping ratio of the mechanical system $b = 0.1$ Nms,
- electromotive force constant $K = 0.01$ Nm/Amp,
- electric resistance $R = 1$ ohm,
- electric inductance $L = 0.5$ H.

Note that $n = 2$. Now, we check the controllability property of the system. The corresponding controllability matrix, according to (6), is given by

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}.$$

Since $\det Q_c = -4 \neq 0$, we get that $\text{rank} Q_c = 2$ and therefore system (8) is fully controllable.

Next, we check the observability property of the system. Again, the observability matrix, according to (7), is given by

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix}.$$

Since $\det Q_c = 1 \neq 0$, we get that $\text{rank} Q_c = 2$ and therefore system (8) is fully observable.

In order to check whether the model of the motor is stable or not, we calculate the eigenvalues of the matrix A.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -10 & 1 \\ 0.02 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \lambda^2 + 12\lambda + 20.02.$$

The solutions to this equation are the eigenvalues $\lambda_1 = -9.9975$ and $\lambda_2 = -2.0025$. Since $\Re(\lambda_1) < 0$ and $\Re(\lambda_2) < 0$, we may conclude that system (8) is stable.

State-space representations of transfer-function systems

Many techniques are available for obtaining state-space representations from the transfer-function description of a system.

Controllable canonical form: The following state-space representation is called a controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \cdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Observable canonical form: The following state-space representation is called an observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Example 2: (Example 9-1, K. Ogata) Consider the system given by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2 + 3s + 2}.$$

Obtain state-space representations in the controllable canonical form and observable canonical form.

From definition of the transfer function (5), we can calculate that $b_0 = 0, b_1 = 1, b_2 = 3, a_1 = 3, a_2 = 2$. Hence,

Controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable canonical form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}3\\ 1\end{bmatrix}u$$
$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Remark 1. *Note that the transfer function can be obtained from state-space representation of continuous-time system as follows $H(s) = C(sI - A)^{-1}B + D$. In a similar manner it can be found for discrete-time systems as well.*

Exercises

Example 3: Check controllability, observability and stability properties of the system. Find input-output forms using state elimination algorithm. Find transfer functions from the state-space and input-output representations and compare obtained results.

1. Continuous-time case:

   (a)
   $$A = \begin{bmatrix}10 & 0\\ -2 & 3\end{bmatrix}, \quad B = \begin{bmatrix}1\\ -2\end{bmatrix}, \quad C = I_{2\times 2}, \quad D = 0_{2\times 1};$$

   (b)
   $$A = \begin{bmatrix}-0.02 & 0 & 0\\ 0.01 & -0.01 & 0\\ 0 & 1 & 0\end{bmatrix}, \; B = \begin{bmatrix}1 & 0\\ 0 & 0.01\\ 0 & 0\end{bmatrix}, \; C = \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \; D = 0_{1\times 2}.$$

2. Discrete-time case:

   (a)
   $$A = \begin{bmatrix}0 & 3\\ 1 & -2\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad C = I_{2\times 2}, \quad D = 0_{2\times 1};$$

   (b)
   $$A = \begin{bmatrix}1 & 1\\ -\frac{1}{4} & 0\end{bmatrix}, \quad B = \begin{bmatrix}1\\ 1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 0\end{bmatrix}, \quad D = 0.$$

Example 4: Given system described by the state equations

$$\begin{aligned}
\dot{x}_1 &= -3x_1 + x_2 + 2x_3 + u_2\\
\dot{x}_2 &= -x_2\\
\dot{x}_3 &= 2x_2 - 4x_3 + u_1\\
y_1 &= 4x_1 + 2x_2\\
y_2 &= 2x_2 + x_3\\
y_3 &= x_3 + u_1
\end{aligned}$$

Check controllability, observability and stability properties of the system.

Example 5: Given system described by the state equations

$$\begin{aligned} x_1(t+1) &= x_2(t) + u(t) \\ x_2(t+1) &= 2x_1(t) + 3x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

Check controllability, observability and stability properties of the system.

Example 6: Given state-space model of the inverted pendulum on a cart

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

in which $x = \begin{bmatrix} \theta & \dot{\theta} & d & \dot{d} \end{bmatrix}^T$, θ (rad) is angle of pendulum, d (m) is position of a cart with respect to the origin, u (N) is force of motion (model input),

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{M+m}{M \cdot l} g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m}{M} g & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\dfrac{1}{M \cdot l} \\ 0 \\ \dfrac{1}{M} \end{bmatrix}, \quad C = I_{4\times4}, \quad D = 0_{4\times4}$$

and parameters are defined as: mass of cart $M = 5$ kg, mass of pendulum $m = 0.5$ kg, distance to pendulum center of mass $l = 0.5$ m, acceleration of gravity $g = 9.81$ m/s^2. Check controllability, observability and stability properties of the system.

Problems

9.1: Check controllability, observability and stability properties of the system. Find transfer functions from the state-space and input-output representations and compare obtained results.

1. Continuous-time case:

   (a)

   $$A = \begin{bmatrix} -1 & 2 \\ 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = I_{2\times2}, \quad D = 0_{2\times1};$$

   (b)

   $$A = \begin{bmatrix} 3 & -7 & 2 \\ 6 & -8 & 0 \\ 3 & 3 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 4 & -1 \end{bmatrix}, \quad D = 0.$$

2. Discrete-time case:

(a)

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = I_{2\times 2}, \quad D = 0_{2\times 1};$$

(b)

$$A = \begin{bmatrix} -5 & 5 & -4 \\ -2 & 0 & -2 \\ -1 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 4 & -3 \end{bmatrix}, \quad D = 0.$$

9.2: Given an input-output continuous-time system

$$y^{(4)} = a_3 y^{(3)} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y + b_2 \ddot{u} + b_1 \dot{u} + b_0 u.$$

Find state-space realizations in controllable and observable canonical forms.

9.3: The Newton's laws of motion for an object moving horizontally on a plane and attached to a wall with a spring

$$m\ddot{y} = u - a_1 \dot{y} - a_0 y,$$

where y is position, $\dot{y}$ is velocity, $\ddot{y}$ is acceleration, u is applied force, a_1 is the viscous friction coefficient, a_0 is the spring constant, m is the mass of the object. Derive the minimal state-space realization of the system and find for which values of a_1, a_0 the system is controllable and observable.

9.4: Consider the "ball and beam" system, whose input u is the angle and whose output y is the ball position. The input-output equation of the system is

$$\ddot{y} = \frac{mR^2}{J + mR^2} \left(y\dot{u}^2 - g \sin(u) \right),$$

where the constant parameters J, R, m represent, respectively, the inertia, radius and mass of the ball, and g is the gravitational constant. Prove that the system is not realizable in the minimal state-space form.

9.5: Consider the second-order bilinear system described by the i/o equation

$$\ddot{y} = a_1 \dot{y} + a_2 y + b_1 \dot{u} + b_2 u + c_{11} \dot{y} \dot{u} + c_{12} \dot{y} u + c_{21} y \dot{u} + c_{22} y u.$$

Find necessary and sufficient realizability conditions in terms of system parameters a_i, b_j, c_{ij} for $i = 1, 2$ and $j = 1, 2$.

9.6: Consider the second-order bilinear system described by the i/o equation[2]

$$y^{++} = a_1 y^+ + a_2 y + b_1 u^+ + b_2 u + c_{11} y^+ u^+ + c_{12} y^+ u + c_{21} y u^+ + c_{22} y u$$

Find necessary and sufficient realizability conditions in terms of system parameters a_i, b_j, c_{ij} for $i = 1, 2$ and $j = 1, 2$.

[2]The abridged notation is used, i.e. $y^+ := y(t+1)$, $u^+ := u(t+1)$, etc.

Answers to problems

1. (a) The system is controllable, observable, and stable. Matrix of transfer functions is
$$H(s) = \begin{pmatrix} \frac{s+8}{s^2+5s+4} \\ \frac{2}{s+4} \end{pmatrix}$$
(b) The system is not controllable, not observable, and unstable. Transfer function is $H(s) = 0$.

2. (a) The system is not controllable, observable, and unstable. Matrix of transfer functions is
$$H(z) = \begin{pmatrix} \frac{z+2}{z^2+3z+2} \\ \frac{z+2}{z^2+3z+2} \end{pmatrix}$$
(b) The system is not controllable, not observable, and unstable. Transfer function is
$$H(z) = \frac{2z^2 + 12z + 16}{z^3 + 7z^2 + 14z + 8}.$$

3. The system is both controllable and observable for any values of a_0 and a_1.

4. The proof can be found in J. Belikov, Ü. Kotta, and M. Tõnso. Minimal realization of nonlinear MIMO equations in state-space form: Polynomial approach. In *The 50th IEEE Conference on Decision and Control and European Control Conference*, pages 7735-7740, Orlando, FL, USA, December 2011.

5. The second-order continuous-time bilinear system is always realizable in the classical state-space form.

6. The second-order discrete-time bilinear system is realizable in the classical state-space form if and only if either one of the following conditions is satisfied: $c_{21} = 0$ or $b_2 = c_{12} = c_{22} = 0$.