ISS0031 Modeling and Identification

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Intro

One of the most important and successful applications of quantitative analysis to solving business problems has been in the physical distribution of products, commonly referred to as **transportation problems**.

The **purpose** is to **minimize the cost** of shipping goods from one location to another so that the **needs** of each arrival area are **met** and every shipping location operates within its **capacity**.

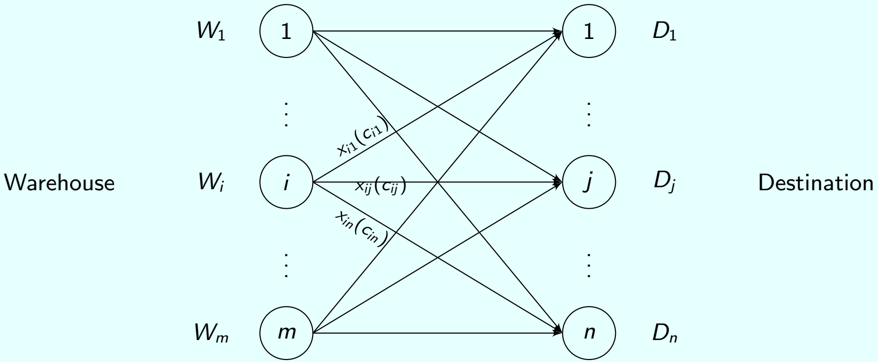
Transportation problems deal with the determination of a minimum-cost plan for transporting a commodity from a number of sources to a number of destinations. To be more specific:

- Let there be m warehouses $W_1, \ldots, W_m$ that have the commodity and n destinations (or consumers) $D_1, \ldots, D_n$ that demand the commodity.
- At the ith warehouse, $i = 1, 2, \ldots, m$, there are a_i units of the commodity available.
- The demand at the jth destination, $j = 1, 2, \ldots, n$, is denoted by b_j.
- The cost of transporting one unit of the commodity from the ith warehouse to the jth destination (route $W_i D_j$) is c_{ij}.
- Let x_{ij} be the numbers of the commodity that are being transported from the ith warehouse to the jth destination.

Problem

Our **problem** is to determine those x_{ij} that will minimize the overall transportation cost. An optimal solution x_{ij} to the problem is called a **transportation plan**.

Transportation problem



The cost of transportation from W_i $(i = 1, \dots, m)$ to D_j $(j = 1, \dots, n)$ will be equal to

$$z = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \to \min .$$

Note that it is not possible to export from the warehouse W_i more than a_i:

$$\begin{aligned} x_{11} + x_{12} + \cdots + x_{1n} &\leq a_1 \\ x_{21} + x_{22} + \cdots + x_{2n} &\leq a_2 \\ &\vdots \\ x_{m1} + x_{m2} + \cdots + x_{mn} &\leq a_m \end{aligned}$$

or shortly

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots, m.$$

Note that the consumer at destination D_j needs b_j commodity or more:

$$\begin{aligned} x_{11} + x_{21} + \cdots + x_{m1} &\geq b_1 \\ x_{12} + x_{22} + \cdots + x_{m2} &\geq b_2 \\ &\vdots \\ x_{1n} + x_{2n} + \cdots + x_{mn} &\geq b_n \end{aligned}$$

or shortly

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n.$$

With the help of the above information we can construct the following table

Warehouse	Destination				Reserve
	D_1	D_2	$\cdots$	D_n	
W_1	c_{11}	c_{12}	$\cdots$	c_{1n}	a_1
W_2	c_{21}	c_{22}	$\cdots$	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	a_m
Requirement	b_1	b_2	$\cdots$	b_n	

Denote by $b = b_1 + b_2 + \cdots + b_n$ the total requirement of commodities and by $a = a_1 + a_2 + \cdots + a_m$ the total amount of available commodities.

Theorem

Transportation problem is **solvable** *if and only if $b \leq a$.*

Remark

If $b > a$, then the transportation problem is **not solvable**. *In this case one has to solve non-mathematical problem either to increase reserve of commodity, or to decrease requirements.*

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Definition

If $a = b$, then transportation problem is called **balanced**.

Remark

For the balanced transportation problem constraints are of the form $\sum_{j=1}^{n} x_{ij} = a_i$ *and* $\sum_{i=1}^{m} x_{ij} = b_j$, *respectively.*

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If the transportation problem is not in the balanced form, i.e. $b < a$, *then one may introduce a fictive destination* D_f *with requirement* $b_f = a - b$, *getting the problem in the balanced form.*

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Consider the transportation problem given by the following table

Warehouse	Destination				Reserve
	D_1	D_2	D_3	D_4	
W_1	3	1	0	4	15
W_2	1	2	5	2	20
W_3	3	8	11	0	25
Requirement	10	10	15	20	

Let us introduce the fictive destination D_f in which a consumer needs 5 units of goods. Since this destination is fictive, the transportation costs can be taken equal to zero.

Warehouse	Destination					Reserve
	D_1	D_2	D_3	D_4	D_f	
W_1	3	1	0	4	0	15
W_2	1	2	5	2	0	20
W_3	3	8	11	0	0	25
Requirement	10	10	15	20	5	

They can be written as the following linear programming problem

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

subject to constraints

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i && 1 \le i \le m, \\ \sum_{i=1}^{m} x_{ij} &= b_j && 1 \le j \le n, \\ x_{ij} &\ge 0 && 1 \le i \le m, 1 \le j \le n, \end{aligned}$$

where $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Using the vector notations

$$
\begin{aligned}
x &= \left[x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}\right]^{\mathrm{T}}, \\
c &= \left[c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{2n}, \ldots, c_{m1}, \ldots, c_{mn}\right]^{\mathrm{T}}, \\
b &= \left[a_1, \ldots, a_m, b_1, \ldots, b_n\right]^{\mathrm{T}},
\end{aligned}
$$

the transportation model can be rewritten using matrix notation

$$
\begin{aligned}
z = c^{\mathrm{T}}x &\to \min \\
Ax &= b \\
x &\geq 0
\end{aligned}
$$

Transportation problems are linear programming problems and can be solved by the Simplex method. Notice that there are mn variables but only $m+n$ equations. To initiate the Simplex method, we have to add $m+n$ more artificial variables and solving the problem by the Simplex method seems to be a very tedious task even for moderate values of m and n. Therefore, because of practical significance and the special structure of the transportation problem we can solve it with a faster, more economical algorithm than simplex.

Preliminary step:
We have to check whether the stated problem is solvable or not.

The starting basic feasible solution: northwest-corner method
We have to distribute the available units in rows and column in such a way that the sum will remain the same. We have to follow the steps given below.

1. Select the north west (upper left-hand) corner (cell) of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e. $\min(a_1, b_1)$.
2. Adjust the supply and demand numbers in the respective rows and columns.
3. If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.
4. If the supply for the first row is exhausted, then move down to the first cell in the second row.
5. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
6. Continue the process until all supply and demand values are exhausted.

There are 3 warehouses W_i for $i = 1, \ldots, 3$ with commodity of the same type in amount of $a_1 = 8$, $a_2 = 10$, $a_3 = 20$ units, respectively, and there are 4 destinations (consumers) D_j for $j = 1, \ldots, 4$ who want to receive at least $b_1 = 6$, $b_2 = 8$, $b_3 = 9$, $b_4 = 15$ units of the commodity, respectively. The cost of transporting one unit of the commodity from warehouse W_i to consumer D_j together with available information are summarized in the following table:

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2	3	5	1	8
W_2	7	3	4	6	10
W_3	4	1	7	2	20
Requirement	6	8	9	15	38

One may see that the problem is balanced, since $\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 38$.

Start allocations from north-west corner, i.e. from (1,1) position.
Here $\min(a_1, b_1) = \min(8, 6) = 6$ units.
Therefore, the maximum possible units that can be allocated to this position is 6.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3}	^{5}	^{1}	$8-6=2$
W_2	^{7} $\times$	^{3}	^{4}	^{6}	10
W_3	^{4} $\times$	^{1}	^{7}	^{2}	20
Requirement	$6-6=0$	8	9	15	38

After completion of the previous step, come across the position (1,2).
Here $\min(8-6, 8) = 2$ units can be allocated to this position.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2 **6**	3 **2**	5 $\times$	1 $\times$	$2-2=0$
W_2	7 $\times$	3	4	6	10
W_3	4 $\times$	1	7	2	20
Requirement	0	$8-2=6$	9	15	30

Now, we go to the second row, here the position (2,1) is already been struck off, so consider the position (2,2).
Here $\min(10, 8-2) = 6$ units can be allocated to this position.
This completes the allocations in second column so strike off the position (3,2).

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2 **6**	3 **2**	5 $\times$	1 $\times$	0
W_2	7 $\times$	3 **6**	4	6	$10-6=4$
W_3	4 $\times$	1 $\times$	7	2	20
Requirement	0	0	9	15	24

Again consider the position (2,3).
Here, $\min(10-6,9)=4$ units can be allocated to this position.
This completes the allocations in second row so struck off the position (2,4).

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2 **6**	3 **2**	5 ×	1 ×	0
W_2	7 ×	3 **6**	4 **4**	6 ×	0
W_3	4 ×	1 ×	7	2	20
Requirement	0	0	$9-4=5$	15	20

In the third row, positions (3,1) and (3,2) are already been struck off so consider the position (3,3) and allocate it the maximum possible units, i.e. $\min(20, 9-4) = 5$ units. Finally, allocate the remaining units to the position (3,4), i.e. 15 units to this position. Keeping in mind all the allocations done in the above method complete the table as follows.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3} **2**	^{5} $\times$	^{1} $\times$	8
W_2	^{7} $\times$	^{3} **6**	^{4} **4**	^{6} $\times$	10
W_3	^{4} $\times$	^{1} $\times$	^{7} **5**	^{2} **15**	20
Requirement	6	8	9	15	38

From the above table we can see that the starting basic feasible solution is $x_{11} = 6$, $x_{12} = 2$, $x_{21} = 6$, $x_{22} = 4$, $x_{32} = 5$, $x_{34} = 15$, and the other variables in the table are $x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$. Therefore, the cost of transportation can be calculated as

$$\begin{aligned} z = c_{11}x_{11} + c_{12}x_{12} + c_{22}x_{22} + c_{23}x_{23} + c_{33}x_{33} + c_{34}x_{34} = \\ = 12 + 6 + 18 + 16 + 35 + 30 = 117. \end{aligned}$$

Step 0. Assume that preliminary step is accomplished and starting feasible solution is found.

Step 1. Find the multipliers u_i, $i = 1, \dots, m$ and v_j, $j = 1, \dots, n$ from the relations

$$u_i + v_j = c_{ij}$$

for all (i, j)-cells containing basic variables. Since there are $m + n - 1$ basic variables, we get the same number of equations. However, there are $m + n$ unknown variables u_i and v_j. Therefore, one of the variables may be fixed, say equal to zero (for example $v_1 = 0$), and the equations may be used to solve for the other variables. Some of the u_i or v_j may turn out to be negative, but this does not matter. Find the indirect transportation costs as

$$\hat{c}_{ij} = u_i + v_j$$

for (i, j)-cells containing non-basic variables.

Step 2. Calculate

$$\varphi = \max(\hat{c}_{ij} - c_{ij}).$$

Check the **optimality criteria**, which is $\varphi = 0$. If it is satisfied, then the obtained transportation plan is optimal. Otherwise, the plan can be improved. It means that we have to redistribute some amount of the commodity, say θ, which has to be put to the cell for which the difference $\hat{c}_{ij} - c_{ij}$ is maximal. However, if we add θ to that cell, we must subtract and add θ to other cells containing basic variables to keep the constraints (requirements vs. reserve) satisfied. We choose θ as large as possible, bearing in mind that negative shipments are not allowed. It means that at least one of the basic variables is put, or remains at, 0.

Step 3. Repeat Steps 1 and 2 until the optimality criteria is satisfied.

Recall the transportation table is

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	$^{2}\ 6$	$^{3}\ 2$	^{5}	^{1}	8
W_2	^{7}	$^{3}\ 6$	$^{4}\ 4$	^{6}	10
W_3	^{4}	^{1}	$^{7}\ 5$	$^{2}\ 15$	20
Requirement	6	8	9	15	38

According to the algorithm presented above we can construct the following system of equations and solve it for u_i, $i = 1, \dots, 3$ and v_j, $j = 1, \dots, 4$

$$\begin{cases} u_1 + v_1 = 2 \\ u_1 + v_2 = 3 \\ u_2 + v_2 = 3 \\ u_2 + v_3 = 4 \\ u_3 + v_3 = 7 \\ u_3 + v_4 = 2 \end{cases}$$

By assigning $v_1 = 0$, we get $u_1 = 2, u_2 = 2, u_3 = 5$ and $v_1 = 0, v_2 = 1, v_3 = 2, v_4 = -3$.

Now, we can add additional column and row and rewrite the transportation table as follows:

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	^{2} 6	^{3} 2	^{5}	^{1}	8	2
W_2	^{7}	^{3} 6	^{4} 4	^{6}	10	2
W_3	^{4}	^{1}	^{7} 5	^{2} 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Next, we calculate the indirect transportation costs as

$$\hat{c}_{ij} = u_i + v_j$$

for (i, j)-cells containing non-basic variables. Note that in the following table $\hat{c}_{ij}$ is placed in the **left-down** corner of the cell.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	^{2} 6	^{3} 2	^{5} 4	^{1} -1	8	2
W_2	^{7} 2	^{3} 6	^{4} 4	^{6} -1	10	2
W_3	^{4} 5	^{1} 6	^{7} 5	^{2} 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Transportation problem

After that, we calculate the difference between the indirect and actual transportation costs as

$$\hat{c}_{ij} - c_{ij}.$$

Note that in the following table the difference is placed in the **right-upper** corner of the cell.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	^{2} 6	^{3} 2	5 -1 4	1 -2 -1	8	2
W_2	7 -5 2	^{3} 6	^{4} 4	6 -7 -1	10	2
W_3	4 1 5	1 5 6	^{7} 5	^{2} 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Transportation problem

Now, we can easily see that $\varphi = \max(\hat{c}_{32} - c_{32}) = 5$.

Since the optimality condition **is not satisfied**, the transportation plan can be improved.

It means that we have to redistribute some amount of the commodity. For that purpose we add θ to the cell (3,2). Since we added θ to that cell, we must subtract it from cells (2,2) and (3,3), respectively. Finally, we have to add θ to the cell (2,3) to keep the constraints satisfied.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	^{2} 6	^{3} 2	^{5} -1 4	^{1} -2 -1	8	2
W_2	^{7} -5 2	^{3} $6-\theta$	^{4} $4+\theta$	^{6} -7 -1	10	2
W_3	^{4} 1 5	^{1} θ ^{5} 6	^{7} $5-\theta$	^{2} 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Doing this way we can see that

$$\theta = \max(6-\theta, 4+\theta, 5-\theta) = 5.$$

After modifying the transportation plan the new table becomes:

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2 6	3 2	5	1	8
W_2	7	3 1	4 9	6	10
W_3	4	1 5	7	2 15	20
Requirement	6	8	9	15	38

From the above table we can see that the cost of transportation can be calculated as

$$z = c_{11}x_{11} + c_{12}x_{12} + c_{22}x_{22} + c_{23}x_{23} + c_{32}x_{32} + c_{34}x_{34} =$$
$$= 12 + 6 + 3 + 36 + 5 + 30 = 107.$$

Repeat Steps 1-3 of the algorithm to get the following tables:

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 $2-\theta$	5 -1 4	1 θ 3 4	8	2
W_2	7 -5 2	3 1	4 9	6 -2 4	10	2
W_3	4 -4 0	1 $5+\theta$	7 -5 2	2 $15-\theta$	20	0
Requirement	6	8	9	15	38	
v_j	0	1	2	2		

for which $\theta = 2$.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 -3 0	5 -4 1	1 2	8	2
W_2	7 -2 5	3 1	4 9	6 -2 4	10	5
W_3	4 -1 3	1 7	7 -5 2	2 13	20	3
Requirement	6	8	9	15	38	
v_j	0	-2	-1	-1		

Observe that the optimality criteria is **satisfied**.

The minimal transportation cost is therefore

$$z = c_{11}x_{11} + c_{14}x_{14} + c_{22}x_{22} + c_{23}x_{23} + c_{32}x_{32} + c_{34}x_{34} =$$

$$= 12 + 2 + 3 + 36 + 7 + 26 = 86.$$