ISS0031 Modeling and Identification

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Motivating example

Let us consider a simple problem in two variables x_1 and x_2. Find x_1 and x_2 which satisfy the following equations

$$x_1 + x_2 = 5$$
$$2x_1 + 3x_2 = 7$$

Solving these equations, we get $x_1 = 8$ and $x_2 = -3$.

Question

What happens when the number of equations and variables will be **greater** or **less**?

Let n be the number of unknowns and m be the number of equations. A linear system may behave in any one of three possible ways:

1. If $m < n$, then a system has infinitely many solutions (underdetermined system).
2. If $m = n$, then a system has a single unique solution.
3. If $m > n$, then a system has no solutions (overdetermined system).

Question

If relations are in the form of inequalities, can we find a **solution** for such a system?

Whenever the analysis of a problem leads to minimizing or maximizing a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution** of the linear programming problem.

To solve a linear programming problem involving two variables by the graphical method, use the following steps

1. Formulate the linear programming problem.
2. Graph the constraints inequalities.
3. Identify the feasible region which satisfies all the constraints simultaneously. For "less than or equal to" constraints the region is generally below the lines and "for greater than or equal to" constraints, the region is above the lines.
4. Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.
5. Evaluate the objective function at each of the vertex (corner point).
6. Identify the optimum value of the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, if an optimal solution exists, then it will occur at a vertex.)

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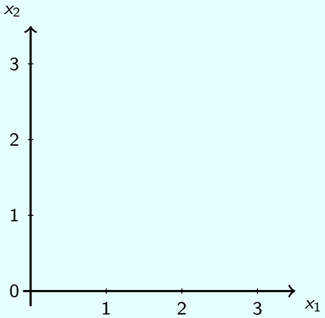
- If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions.
- If the problem has more than one solution, then at least one of them must occur at a vertex of the set of feasible solutions.
- If none of the feasible solutions maximizes (or minimizes) the objective function, or if there are no feasible solutions, then the linear programming problem has no solutions.

Consider the following linear programming problem

$$z = x_1 + 2x_2 \to \max(\min)$$
$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0, x_2 \geq 0.$$

$$x_1 + x_2 = 1 \Rightarrow x_2 = 1 - x_1$$

	x_1	x_2	z



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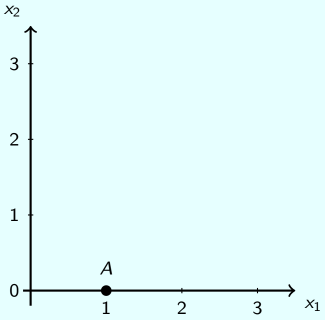

	x_1	x_2	z
A	1	0	1

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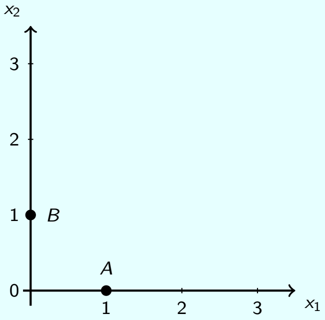

	x_1	x_2	z
A	1	0	1
B	0	1	2

Consider the following linear programming problem

$$z = x_1 + 2x_2 \to \max(\min)$$
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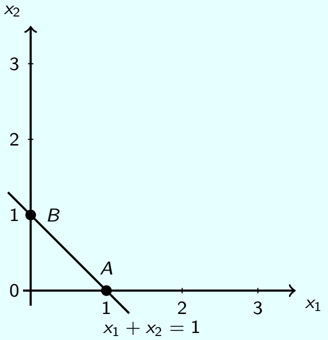

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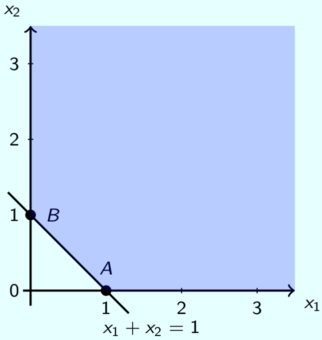

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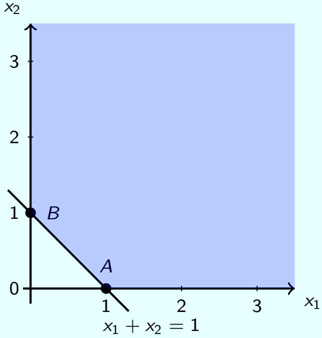

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A	1	0	1
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- **Minimum:** $z_{\min}(A) = 1$.

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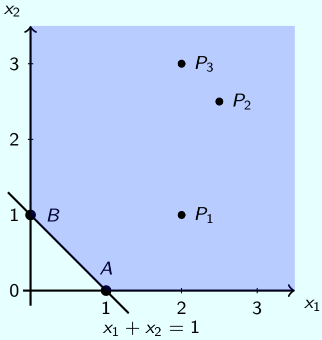
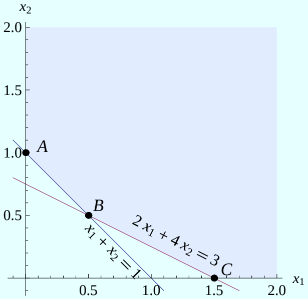


	x_1	x_2	z
A	1	0	1
B	0	1	2

- **Minimum:** $z_{\min}(A) = 1$.
- **Maximum:** $z(P_1) = 4$, $z(P_2) = 7.5$, $z(P_3) = 8$, ... The region, determined by the constraints, is **unbounded**.
  $\Rightarrow$ no feasible point that will make z largest.
  $\Rightarrow$ linear programming problem has **no solution** in case of maximizing the objective function.

Consider the following linear programming problem

$$z = x_1 + 2x_2 \to \min$$
$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0, x_2 \geq 0$$



- ✓ The shaded region is a set of feasible solutions.
- ✓ **Optimal solution**: check $z(A) = 2$, $z(B) = 1.5$, and $z(C) = 1.5$. $\Rightarrow$ the objective function has a minimum value not only at the vertices B and C, it also has a minimum value at any point on the line segment connecting these two vertices, i.e,

$$x_1 = \frac{3 - 2\alpha}{2}, x_2 = 0.5\alpha, 0 \leq \alpha \leq 1.$$

- ✓ The linear programming problem has **infinitely many solutions** and two of them occur at the vertices.

Consider the following linear programming problem

$$
\begin{aligned}
z = x_1 + 2x_2 &\to \min \\
x_1 + x_2 &\geq 1 \\
2x_1 + 4x_2 &\geq 3 \\
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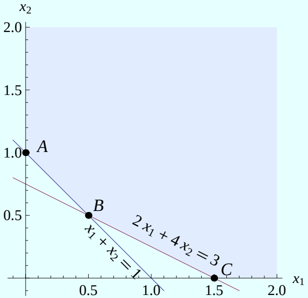

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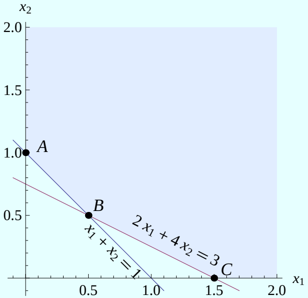

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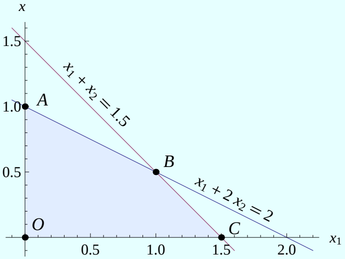
Consider the following linear programming problem

$$
\begin{aligned}
z = 2x_1 + x_2 &\to \max(\min)\\
x_1 + 2x_2 &\leq 2\\
x_1 + x_2 &\leq 1.5\\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$



- ✓ The shaded region $OABC$ is the feasible region.
- ✓ **Optimal solution**: check the vertices $z(A) = 1$, $z(B) = 2.5$, $z(C) = 3$, and $z(O) = 0$.
- ✓ If we take any other value from the feasible region we see that still the maximum value is 3 obtained at the vertex $C(1.5, 0)$ of the feasible region. The same arguments hold for the minimum value $z_{\min}(0, 0) = 0$.

Exercise I

$$
\begin{aligned}
z = 3x_1 + 2x_2 &\to \max \\
x_1 + 2x_2 &\leq 4 \\
x_1 - x_2 &\leq 1 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

Exercise II

$$
\begin{aligned}
z = 3x_1 + 4x_2 &\to \max \\
x_1 + x_2 &\leq 40 \\
x_1 + 2x_2 &\leq 60 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

Exercise III

$$
\begin{aligned}
z = 3x_1 + 4x_2 &\to \max \\
x_1 + x_2 &\leq 40 \\
x_1 + 2x_2 &\leq 60 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

See you next week!

Hopefully.