ISS0031 Modeling and Identification

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2 variables $\rightarrow$ geometric approach

What if we have **more** than 2 variables?

Consider the following linear programming problem:

$$
\begin{aligned}
z = 3x_1 + 4x_2 &\to \max \\
2x_1 + 4x_2 &\leq 120 \\
2x_1 + 2x_2 &\leq 80 \\
x_1 \geq 0, x_2 &\geq 0.
\end{aligned}
$$

The procedure for solving the given problem is illustrated in the following steps:

Step 1: Standard form of a maximum problem
A linear programming problem in which the objective function is to be maximized is referred to as a maximum linear programming problem. Such problems are said to be in standard form if the following **conditions** are satisfied:

- all the variables are nonnegative;
- all the other constraints are written as a linear expression, that is, less than or equal to a positive constant.

Summary:

$$z = CX \to \max$$
$$AX \leq B$$
$$X \geq 0$$

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This particular example is a maximum problem:

- it contains two variables x_1 and x_2;
- both variables are non-negative;
- constraints are each written as linear expression less than or equal to a positive constant.

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In order to solve the maximum problem by simplex method, we need to do the following first:

- introduction of **slack variables**;
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Constraints are linear expressions less than or equal to some positive constants. That means there is a slack between the left and right sides of the inequalities.

In order to take up the slack between the left and right sides of the problem, let us introduce the slack variables s_1 and s_2 which are greater than or equal to zero, such that

$$2x_1 + 4x_2 + s_1 = 120$$
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The objective function z can be rewritten as $z - 3x_1 - 4x_2 = 0$. System of three equations in five unknowns:

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Goal

Find the particular solution (x_1, x_2, s_1, s_2, z) that gives the largest possible value for z.

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Find the particular solution (x_1, x_2, s_1, s_2, z) that gives the largest possible value for z.

The initial (starting) simplex table (matrix):

x_1	x_2	s_1	s_2	z	b	variables
2	4	1	0	0	120	s_1
2	2	0	1	0	80	s_2
-3	-4	0	0	1	0	z

Notice that the coefficients of the objective function are arranged in the bottom row which is called the **objective row**.

The simplex method consists of **pivoting** from one table to another until the optimal solution is found.

Pivoting: To pivot a matrix about a given element, called the pivot element, is to apply row operations so that the pivot element is replaced by 1 and all other entries in the same column (called pivot column) become 0. More specifically, in the pivot row, divide each entry by the pivot element (we assume it is not 0). Obtain 0 elsewhere in the pivot column by performing row operations.

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Pivot element: The pivot element for the Simplex method is found using the following rules:

- The pivot column is selected by locating the **most negative** entry in the objective row. If all the **entries** in this **column** are **negative**, the problem is **unbounded** and there is no solution.
- Divide each entry in the last column by the corresponding entry (from the same row) in the pivot column. (Ignore the rows in which the pivot column entry is less than or equal to 0). The row in which the smallest positive ratio is obtained is the pivot row.

The pivot element is the entry at the **intersection** of the pivot row and the pivot column.

- $--4$ is the most negative entry in the objective row $\Rightarrow$ the second column is the pivot column.
- Dividing all the entries in the fifth column by the corresponding entry in the second column, we get 30 as the smallest ratio $\Rightarrow$ the first row is the pivot row.

$$a_{21} = 4 \text{ is the pivot element.}$$

x_1	x_2	s_1	s_2	z	b	variables	ratio
2	**4**	1	0	0	120	s_1	$\frac{120}{4} = 30$
2	2	0	1	0	80	s_2	$\frac{80}{2} = 40$
-3	-4	0	0	1	0	z	

Divide R_1 (the first row) by 4 and then apply the operations

$$R_2 - 2R_1 \text{ and } R_3 + 4R_1.$$

The new table becomes

x_1	x_2	s_1	s_2	z	b	variables	ratio
$\frac{1}{2}$	1	$\frac{1}{4}$	0	0	30	x_2	60
1	0	$-\frac{1}{2}$	1	0	20	s_2	20
-1	0	1	0	1	120	z	

Now -1 is the most negative entry in the objective row, so the first column is the pivot column. On dividing all the entries in the the fifth column by the corresponding entry in the first column, we get 20 as the smallest ratio, so the second row is the pivot row. Hence, $a_{21} = 1$ is the pivot element. This is marked by a bold font in the second simplex table.

Next, apply the operations $R_1 - \frac{1}{2}R_2$ and $R_3 + R_2$. The new table becomes

x_1	x_2	s_1	s_2	z	b	variables
0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	20	x_2
1	0	$-\frac{1}{2}$	1	0	20	x_1
0	0	$\frac{1}{2}$	1	1	140	z

In the above table we notice that there are **no** negative entries in the objective row.

Conclusion

The optimal solution has been found. Therefore, $z_{\max}(20, 20) = 140$.

Note that, in general, a minimum problem can be **changed** to a maximum problem by realizing that in order to minimize z we must maximize $-z$. That is in such cases we multiply the objective function by -1 and convert it into a maximum problem and solve it as discussed above.

Consider the following linear programming problem:

$$
\begin{aligned}
z = x_1 - 3x_2 + 2x_3 &\to \min \\
3x_1 - x_2 + 2x_3 &\le 7 \\
-2x_1 + 4x_2 &\le 12 \\
-4x_1 + 3x_2 + 8x_3 &\le 10 \\
x_1 \ge 0, x_2 \ge 0, x_3 &\ge 0
\end{aligned}
$$

This is a problem of minimization in which all the constraints are written as a linear expression, that is, less than or equal to a positive constant. Therefore, converting the objective function for maximization, we have

$$z' = -x_1 + 3x_2 - 2x_3 \to \max,$$

where $z' = -z$.

Simplex method

After introducing the slack variable the problem can be expressed as

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 + s_1 &= 7 \\
-2x_1 + 4x_2 + s_2 &= 12 \\
-4x_1 + 3x_2 + 8x_3 + s_3 &= 10 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \\
s_1 \geq 0, s_2 \geq 0, s_3 &\geq 0
\end{aligned}
$$

The objective function can be written as $z' + x_1 - 3x_2 + 2x_3 = 0$. The initial simplex table for this system is

x_1	x_2	x_3	s_1	s_2	s_3	z'	b	variables	ratio
3	−1	2	1	0	0	0	7	s_1	
−2	**4**	0	0	1	0	0	12	s_2	$\frac{12}{4} = 3$
−4	3	8	0	0	1	0	10	s_3	$\frac{10}{3}$
1	−3	2	0	0	0	1	0	z	

The pivot column: the smallest entry in the objective row is -3 in the second column.

The pivot row: divide each entry in the fifth column by the corresponding entry in the pivot column and select the smallest non-negative ratio.

$\Rightarrow$ the second row is the pivot row and the pivot element in that row is the element 4.

Dividing R_2 by 4 and applying the operations

$$R_1 + R_2,\ R_3 - 3R_2, \text{ and } R_4 + 3R_2,$$

we get the second simplex table as

x_1	x_2	x_3	s_1	s_2	s_3	z	b	variables
$\mathbf{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	0	10	s_1
$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	0	3	x_2
$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	0	1	s_3
$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	1	9	z

Applying the same procedure, we determine the next pivot element to be $\frac{5}{2}$ in the first column and first row.

Dividing R_1 by $\frac{5}{2}$ and applying the operations

$$R_2 + \frac{1}{2}R_1,\ R_3 + \frac{5}{2}R_1, \text{ and } R_4 + \frac{1}{2}R_1,$$

we get the third simplex table as

x_1	x_2	x_3	s_1	s_2	s_3	z	b	variables
1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	0	4	x_1
0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	0	5	x_2
0	0	10	1	$-\frac{1}{2}$	1	0	11	s_3
0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	1	11	z

In the above table we see that there are **no** negative entries in the objective row.

Conclusion

Hence, the optimal solution is found. Therefore, $z_{\max} = 11$ for $x_1 = 4, x_2 = 5, x_3 = 0, s_1 = 0, s_2 = 0, s_3 = 11$.

Hence, the solution of the original problem is $z_{\min}(4, 5, 0) = -11$.

Consider the following linear programming problem:

$$z = x_1 + 0.5x_2 \to \max$$
$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 3$$
$$x_1 \geq 0, x_2 \geq 0$$

As before, we add slack variables s_1, s_2 and solve the problem by the Simplex method, using table representation.

x_1	x_2	s_1	s_2	z	b	variables	ratio
2	1	1	0	0	4	s_1	$\frac{4}{2} = 2$
1	2	0	1	0	3	s_2	$\frac{3}{1} = 3$
-1	$-\frac{1}{2}$	0	0	1	0	z	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	x_1	
0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	s_2	
0	0	$\frac{1}{2}$	0	1	2	z	

Now, we may see that this is an optimal solution. Interestingly, the coefficient of the nonbasic variable x_2 in the objective row happens to be equal to 0. However, if we increase x_2 (from its current value of 0), this will not effect the value of z. Increasing x_2 produces changes in the other variables, of course, through the equations in rows 1 and 2. In fact, we can pivot to get a different basic solution with the same objective value $z = 2$.

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x_1	x_2	s_1	s_2	z	b	variables	ratio
2	1	1	0	0	4	s_1	$\frac{4}{2} = 2$
1	2	0	1	0	3	s_2	$\frac{3}{1} = 3$
-1	$-\frac{1}{2}$	0	0	1	0	z	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	x_1	
0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	s_2	
0	0	$\frac{1}{2}$	0	1	2	z	

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x_1	x_2	s_1	s_2	z	b	variables
1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	x_1
0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	x_2
0	0	$\frac{1}{2}$	0	1	2	z

Therefore, the optimal solutions can be represented as $z_{\max}(x) = 2$, where

$$x = \alpha p_1 + (1-\alpha)p_2 = \left(\frac{1}{3}\alpha + \frac{5}{3}, (1-\alpha)\frac{2}{3}, 0, \alpha\right)$$

with $p_1 = (2, 0, 0, 1)$, $p_2 = (\frac{5}{3}, \frac{2}{3}, 0, 0)$ and $0 \leq \alpha \leq 1$.

Remark

The linear programming problem has alternative optimal solutions (multiple optimal solutions) if, at least, one of the coefficients of the nonbasic variable in the objective row equals to zero.

Summary of the Simplex method:

Step 1: Add slack variables to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.

Step 2: Write the objective function with all nonzero terms to the left of the equal sign and zero to the right. The variable to be maximized must be positive.

Step 3: Set up the initial simplex tableau by creating an augmented matrix from the equations, placing the equation for the objective function last.

Step 4: Determine a pivot element and use matrix row operations to convert the column containing the pivot element into a unit column.

Step 5: If negative elements still exist in the objective row, repeat **Step 4**. If all elements in the objective row are positive, the process has been completed.

Step 6: When the final matrix has been obtained, determine the final basic solution. This will give the maximum value for the objective function and the values of the variables where this maximum occurs.

Geometric interpretation of the Simplex method: The simplex method always starts at the origin (which is a corner point) and then jumps from a corner point to the neighboring corner point until it reaches the optimal corner point (if bounded). Therefore, at each one of the simplex iterations, we are searching for a better solution among the vertices of a Simplex.

Special cases

$$
\begin{aligned}
z = 4x_1 + 3x_2 &\to \max \\
3x_1 + x_2 &\leq 9 \\
-x_1 + x_2 &\leq 1 \\
x_1 + x_2 &\leq 6 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$