ISS0031 Modeling and Identification

Lecture 1

Introductory examples

Example 1: A phone dealer goes to the wholesale market with 1500 EUR to purchase phones for selling. In the market there are various types of phones available. From quality point of view, he finds that the phone of type P_1 and type P_2 are suitable. The cost price of type P_1 phone is 300 EUR/item and that of type P_2 is 250 EUR/item. He knows that one phone of the type P_1 can be sold for 325 EUR, while phone of the type P_2 can be sold for 265 EUR. Within the available amount of money he would like to make maximum profit. His problem is to find out how many type P_1 and type P_2 phones should be purchased so to get the maximum profit.

The dealer can prepare the following table taking into account all possible combinations of type P_1 and type P_2 phones subject to the limitation on the investment.

P_1	P_2	Investment	Amount after sale	Profit on the investment
0	6	1500	1590	90
1	4	1300	1385	85
2	3	1350	1445	95
3	2	1400	1505	105
4	1	1450	1565	115
5	0	1500	1625	125

Now, the decision leading to maximum profit is clear. Five type P_1 phones should be purchased. Here we have to maximize the profit. Sometimes we come across a problem in which the costs are to be minimized.

Example 2: Two tailors T_1 and T_2 earn 150 EUR and 200 EUR per day, respectively. T_1 can stitch 6 shirts and 4 pants per day, while T_2 can stitch 4 shirts and 7 pants per day. How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost? In this problem we have to minimize the labour cost.

These type of problems of maximization and minimization are called optimization problems.

Mathematical programming problem

Mathematical programming problem can be written in the form

$$\begin{aligned} g_1(x_1, x_2, \ldots, x_n) &\leq 0 \\ g_2(x_1, x_2, \ldots, x_n) &\leq 0 \\ &\vdots \\ g_m(x_1, x_2, \ldots, x_n) &\leq 0 \\ (x_1, x_2, \ldots, x_n) &\in S \subset \mathbb{R}^n \end{aligned}$$

Definition 1. *The function to be maximized*

$$z = f(x_1, x_2, \ldots, x_n) \to \max$$

or minimized

$$z = f(x_1, x_2, \ldots, x_n) \to \min$$

is called the ***objective function****.*

Definition 2. *The limitations on resources which are to be allocated among various competing variables are in the form of equations or inequalities and are called* ***constraints*** *or* ***restrictions****.*

Major subfields of mathematical optimization:

1. Linear programming studies the case in which the objective function is linear and the set of constraints is specified using only linear equalities and inequalities.

2. Nonlinear programming studies the general case in which the objective function or the constraints or both contain nonlinear parts.

3. Integer programming studies linear programs in which some or all variables are constrained to take on integer values.

4. Stochastic programming studies the case in which some of the constraints or parameters depend on random variables.

5. Optimal control theory is a generalization of the calculus of variations.

Linear programming: basic concepts

Definition 3. *A* ***linear programming problem*** *may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints.*

The standard maximum problem can be stated as: Find a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, to maximize

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0.$$

Definition 4. *A vector x for the optimization problem is said to be **feasible** if it satisfies all the constraints.*

Definition 5. *A vector x is **optimal** if it feasible and optimizes the objective function over feasible x.*

Definition 6. *A linear programming problem is said to be **feasible** if there exist a feasible vector x for it; otherwise, it is said to be **infeasible**.*

Lemma 1. *Every linear programming problem is either bounded feasible, unbounded feasible, or infeasible.*

Definition 7. *A **basic solution** is a solution that is obtained by fixing enough variables to be equal to zero, so that the equality constraints have a unique solution.*

A basis solution exists if and only if the columns of corresponding equality constraint form a basis. In other words, a largest possible linearly independent collection.

Definition 8. *A feasible solution to linear programming problem problem which is also the basic solution is called the **basic feasible solution**. Basic feasible solutions are of two types:*

- *Degenerate: if value of at least one basic variable is zero.*
- *Non-degenerate: if all values of basic variables are non-zero and positive.*

Matrix form: Suppose that

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

then linear programming problem can be rewritten in the standard or canonical matrix form as

$$\begin{aligned} z = CX &\to \max(\min) \\ AX &\le B \\ X &\ge 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} z = CX &\to \max(\min) \\ AX &= B \\ X &\ge 0 \end{aligned}$$

Formulation of a linear programming problem: The formulation of linear programming problem as a mathematical model involves the following 3 steps:

1. identify the decision variables to be determined and express them in terms of algebraic symbols such as $x_1, x_2, \dots, x_n$;
2. identify the objective which is to be optimized (maximized or minimized) and express it as a linear function of the above defined decision variables;
3. identify all the limitations in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Solution methods: Once the problem is formulated by setting appropriate objective function and constraints, the next step is to solve it. Solving linear programming problem is nothing but determining the values of decision variables that maximizes or minimizes the given effective measure satisfying all the constraints. There are many methods for solving linear programming problems which are listed below.

1. Graphical method.
2. Analytical method or trial and error method.
3. Simplex method.
4. Big-M method.
5. Two phase simplex method.
6. Dual simplex method.
7. Revised simplex method.

The graphical method is used for solving linear programming problem with two decision variables only. If there are more than two decision variables, then the problem is to be solved using analytical methods.

Common linear programming problems: illustrative examples

Example 3: The Production Planning Problem. A company manufactures two types of products P_1 and P_2 and sells them at a profit of 2 EUR and 3 EUR, respectively. Each product is processed on two machines M_1 and M_2. P_1 requires 1 minute of processing time on M_1 and 2 minutes on M_2, type P_2 requires 1 minute on M_1 and 1 minute on M_2. The machine M_1 is available for not more than 6 hours and 40 minutes, while machine M_2 is available for 10 hours during one working day. The given information in the problem can systematically be arranged in the form of following table:

Machine	Processing time		Available time
	P_1	P_2	
M_1	1	1	400
M_2	2	1	600
Profit	2	3	

The problem is to maximize the profit of the company.

Let x_1 and x_2 be the number of products of type P_1 and P_2, respectively. Since the profit on type P_1 is 2 EUR per product, we get $2x_1$. Similarly, the profit on selling x_2 units of type P_2 will be $3x_2$. Therefore, total profit on selling x_1 units of type P_1 and x_2 units of type P_2 is given by (objective function)

$$z = 2x_1 + 3x_2.$$

Since machine M_1 takes 1 minute time on type P_1 and 1 minute time on type P_2, therefore, the total number of minutes required on machine M_1 is given by $x_1 + x_2$. But the machine M_1 is not available for more than 6 hours and 40 minutes (i.e., 400 minutes). Therefore,

$$x_1 + x_2 \leq 400.$$

Similarly, the total number of minutes required on machine M_2 is given by $2x_1 + x_2$. Also, the machine M_2 is available for 10 hours (i.e., 600 minutes). Therefore,

$$2x_1 + x_2 \leq 600.$$

Since, it is not possible to produce negative quantities, so

$$x_1 \geq 0, x_2 \geq 0.$$

Thus, the problem is to find x_1 and x_2 which maximize the objective function z. The problem can be formally written as

$$\begin{aligned} z = 2x_1 + 3x_2 &\to \max \\ x_1 + x_2 &\leq 400 \\ 2x_1 + x_2 &\leq 600 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 4: The Diet Problem. Let there be three different types of food F_1, F_2, F_3, that supply varying quantities of 2 nutrients N_1, N_2, that are essential to good health. Suppose a person has decided to make an individual plan to improve the health. We know that 400 g and 1 kg are the minimum daily requirements of nutrients N_1 and N_2, respectively. Moreover, the corresponding unit of food F_1, F_2, F_3 costs 2, 4 and 3 EUR, respectively. Finally, we know that

- one unit of food F_1 contains 20 g of nutrient N_1 and 40 g of nutrient N_2;
- one unit of food F_2 contains 25 g of nutrient N_1 and 62 g of nutrient N_2;
- one unit of food F_3 contains 30 g of nutrient N_1 and 75 g of nutrient N_2.

The given information can be arranged in the form of the following table:

Nutrients	Food			Requirement/day
	F_1	F_2	F_3	
N_1	20	25	30	400
N_2	40	62	75	1000
Price	2	4	3	

The problem is to supply the required nutrients at minimum cost.

Let x_i for $i = 1, 2, 3$ be the number of units of food F_i to be purchased per day. The problem can be formally written as

$$\begin{aligned} z = 2x_1 + 4x_2 + 3x_3 \to \min \\ 20x_1 + 25x_2 + 30x_3 \geq 400 \\ 40x_1 + 62x_2 + 75x_3 \geq 1000 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Example 5: The Transportation Problem. There are 3 warehouses W_i for $i = 1, \ldots, 3$ with commodity of the same type in amount of 200, 300, 450 units, respectively, and there are 4 consumers C_j for $j = 1, \ldots, 4$ who want to receive at least 150, 300, 150, 200 units of the commodity, respectively. The cost of transporting one unit of the commodity from warehouse W_i to consumer C_j together with available information are summarized in the following table:

Warehouse	Consumers				Reserve
	C_1	C_2	C_3	C_4	
W_1	3	2	7	1	200
W_2	1	4	5	2	300
W_3	2	7	4	3	450
Requirement	150	300	150	200	

The problem is to meet the consumer requirements at minimum transportation cost.

Let x_{ij} $i = 1, 2, 3$ and $j = 1, \ldots, 4$ be the quantity of the commodity shipped from warehouse W_i to consumer C_j. The total transportation cost is

$$\begin{aligned} z = 3x_{11} + 2x_{12} + 7x_{13} + x_{14} + x_{21} + 4x_{22} + \\ + 5x_{23} + 2x_{24} + 2x_{31} + 7x_{32} + 4x_{33} + 3x_{34} \to \min . \end{aligned}$$

The amount sent from and available at the warehouse W_i lead to the following constraints

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} \leq 200 \\ x_{21} + x_{22} + x_{23} + x_{24} \leq 300 \\ x_{31} + x_{32} + x_{33} + x_{34} \leq 450 \end{aligned}$$

The amount sent to and required by the consumer C_j results in

$$\begin{aligned} x_{11} + x_{21} + x_{31} \geq 150 \\ x_{12} + x_{22} + x_{32} \geq 300 \\ x_{13} + x_{23} + x_{33} \geq 150 \\ x_{14} + x_{24} + x_{34} \geq 200 \end{aligned}$$

It is assumed that we cannot send a negative amount from W_i to C_j

$$x_{ij} \geq 0$$

for $i = 1, 2, 3$ and $j = 1, \ldots, 4$. Therefore, the problem is: minimize z subject to constraints listed above.

Example 6: The Optimal Assignment Problem. There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers x_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus, we get that

$$\sum_{j=1}^{J} x_{ij} \leq 1$$

for $i = 1, \ldots, I$, which reflects the fact that a person cannot spend more than 100% of his time working;

$$\sum_{i=1}^{I} x_{ij} \leq 1$$

$j = 1, \ldots, J$, which means that only one person is allowed on a job at a time;

$$x_{ij} \geq 0$$

for $i = 1, \ldots, I$ and $j = 1, \ldots, J$, which says that no one can work a negative amount of time on any job. We wish to maximize the total value

$$z = \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} x_{ij} \to \max .$$

Problems

1.1: A dealer has 1500 EUR only for a purchase of rice and wheat. A bag of rice costs 150 EUR and a bag of wheat costs 120 EUR. He has a storage capacity of ten bags only and the dealer gets a profit of 11 EUR and 8 EUR per bag of rice and wheat, respectively. Formulate the problem of deciding how many bags of rice and wheat should dealer buy in order to get the maximum profit.

1.2: Mr. Bob's bakery sells bagel and muffins. To bake a dozen bagels Bob needs 5 cups of flour, 2 eggs, and one cup of sugar. To bake a dozen muffins Bob needs 4 cups of flour, 4 eggs and two cups of sugar. Bob can sell bagels in 10 EUR/dozen and muffins in 12 EUR/dozen. Bob has 50 cups of flour, 30 eggs and 20 cups of sugar. Formulate the problem of deciding how many bagels and muffins should Bob bake in order to maximize his revenue.

1.3: A company makes two types of sofas, regular and long, at two locations, one in Tallinn and one in Tartu. The plant in Tallinn has a daily operating budget of

45 000 EUR and can produce at most 300 sofas daily in any combination. It costs 150 EUR to make a regular sofa and 200 EUR to make a long sofa at the Tallinn plant. The Tartu plant has a daily operating budget of 36 000 EUR, can produce at most 250 sofas daily in any combination and makes a regular sofa for 135 EUR and a long sofa for 180 EUR. The company wants to limit production to a maximum of 250 regular sofas and 350 long sofas each day. The company makes a profit of 50 EUR on each regular sofa and 70 EUR on each long sofa. Formulate the problem of deciding how many of each type should be made at each plant in order to maximize profit.

1.4: A small company produces two types of products bacon and cheese and sells them at a profit of 4 EUR/kg and 6 EUR/kg, respectively. A student is trying to decide on lowest cost diet that provides sufficient amount of proteins and fats. He knows that bacon contains 2 units of protein/kg, 5 units of fat/kg and cheese contains 2 units of protein/kg, 3 units of fat/kg. Moreover, for the proper diet student needs to consume 9 units of protein/day and 10 units of fat/day. Formulate the problem of deciding how much student should consume of food to meet the daily norm and the cost of food was minimal.

1.5: There are m different types of food $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients $N_1, \ldots, N_n$, that are essential to good health. Let b_j be the minimum daily requirement of nutrient N_j. Let c_i be the price per unit of food F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

1.6: There are m ports, or production plants P_i for $i = 1, \ldots, m$, that supply a certain commodity, and there are n markets M_j for $j = 1, \ldots, n$ to which this commodity must be shipped. Port P_i possesses an amount a_i of the commodity, and market M_j must receive the amount b_j of the commodity. Let c_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

1.7: There are 5 jobs that have to be given to 5 workers in such a way that each job is performed by only one worker. Since each worker can spend a certain amount of time to perform a certain task, we need to find a distribution of tasks among all workers that the total time was minimal. The table below gives the amount of time required for each worker to perform the corresponding job:

Job	Worker				
	W_1	W_2	W_3	W_4	W_5
J_1	5	4	3	6	4
J_2	3	9	8	8	4
J_3	2	1	4	5	6
J_4	3	4	2	4	3
J_5	2	6	5	3	2

1.8: A company is involved in the production of two items (I_1 and I_2). The resources need to produce I_1 and I_2 are twofold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

Item	Machine time	Craftsman time
I_1	13	20
I_2	19	29

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at 10 EUR per hour worked and craftsman time is costed at 2 EUR per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is 20 EUR for I_1 and 30 EUR for I_2. The company has a specific contract to produce 10 items of I_1 per week for a particular customer. Formulate the problem of deciding how much to produce per week as a linear program.

1.9: A company makes two products (P_1 and P_2) using two machines (M_1 and M_2). Each unit of P_1 that is produced requires 50 minutes processing time on machine M_1 and 30 minutes processing time on machine M_2. Each unit of P_2 that is produced requires 24 minutes processing time on machine M_1 and 33 minutes processing time on machine M_2. At the start of the current week there are 30 units of P_1 and 90 units of P_2 in stock. Available processing time on machine M_1 is forecast to be 40 hours and on machine M_2 is forecast to be 35 hours. The demand for P_1 in the current week is forecast to be 75 units and for P_2 is forecast to be 95 units. Company policy is to maximize the combined sum of the units of P_1 and the units of P_2 in stock at the end of the week. Formulate the problem of deciding how much of each product to make in the current week as a linear program.

1.10: Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and three times the first number plus the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

Answers to problems

1. Let x_1 be rice and x_2 be wheat, then

$$z = 11x_1 + 8x_2 \to \max$$

$$\begin{aligned} 5x_1 + 4x_2 &\leq 50 \\ x_1 + x_2 &\leq 10 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

2. Let x_1 be bagels and x_2 be muffins, then

$$z = 10x_1 + 12x_2 \to \max$$

$$\begin{aligned} 5x_1 + 4x_2 &\leq 50 \\ 2x_1 + 4x_2 &\leq 30 \\ x_1 + 2x_2 &\leq 20 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

3. Let x_1 be regular sofas made in Tallinn, x_2 be long sofas made in Tallinn, x_3 be regular sofas made in Tartu, and x_4 be long sofas made in Tartu, then

$$z = 50x_1 + 70x_2 + 50x_3 + 70x_4 \to \max$$

$$\begin{aligned} 150x_1 + 200x_2 &\leq 45000 && \text{money constraint at Tallinn} \\ x_1 + x_2 &\leq 300 && \text{Tallinn sofa limit} \\ 135x_3 + 180x_4 &\leq 36000 && \text{money constraint at Tartu} \\ x_3 + x_4 &\leq 250 && \text{Tartu sofa limit} \\ x_1 + x_3 &\leq 250 && \text{regular sofa limit} \\ x_2 + x_4 &\leq 350 && \text{long sofa limit} \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 &\geq 0 \end{aligned}$$

4. Let x_1 be bacon and x_2 be cheese, then

$$z = 4x_1 + 6x_2 \to \min$$

$$\begin{aligned} 2x_1 + 2x_2 &\geq 9 \\ 5x_1 + 3x_2 &\geq 10 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

5. Let x_i be the number of units of food F_i to be purchased per day, then

$$z = c_1x_1 + c_2x_2 + \cdots + c_mx_m \to \min$$

$$\begin{aligned} a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m &\geq b_j \\ x_i &\geq 0 \end{aligned}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

6. Let x_{ij} be the quantity of the commodity shipped from port P_i to market M_j, then

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\le a_i \\ \sum_{i=1}^{m} x_{ij} &\ge b_j \\ x_{ij} &\ge 0 \end{aligned}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$.

7. Let x_{ij} be the ith job performed by the jth worker. Moreover, $x_{ij} = 1$ or $x_{ij} = 0$ means that whether the jth worker performs the ith job or not. Then,

$$\begin{aligned} z = 5x_{11} + 4x_{12} + 3x_{13} + 6x_{14} + 4x_{15} + \\ + 3x_{21} + 9x_{22} + 8x_{23} + 8x_{24} + 4x_{25} + \\ + 2x_{31} + x_{32} + 4x_{33} + 5x_{34} + 6x_{35} + \\ + 3x_{41} + 4x_{42} + 2x_{43} + 4x_{44} + 3x_{45} + \\ + 2x_{51} + 6x_{52} + 5x_{53} + 3x_{54} + 2x_{55} \to \min \end{aligned}$$

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} &= 1 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} &= 1 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} &= 1 \\ x_{41} + x_{42} + x_{43} + x_{44} + x_{45} &= 1 \\ x_{51} + x_{52} + x_{53} + x_{54} + x_{55} &= 1 \\ x_{11} + x_{21} + x_{31} + x_{41} + x_{51} &= 1 \\ x_{12} + x_{22} + x_{32} + x_{42} + x_{52} &= 1 \\ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} &= 1 \\ x_{14} + x_{24} + x_{34} + x_{44} + x_{54} &= 1 \\ x_{15} + x_{25} + x_{35} + x_{45} + x_{55} &= 1 \\ x_{ij} &\in \{0, 1\} \qquad \text{for } i, j = 1, \dots, 5 \end{aligned}$$

8. Let x_1 be the number of items of I_1 and x_2 be the number of items of I_2, then

$$\begin{aligned} z = 20x_1 + 30x_2 - 10(13x_1 + 19x_2)/60 - \\ - 2(20x_1 + 29x_2)/60 = 17.1667x_1 + 25.8667x_2 \to \max \end{aligned}$$

$$\begin{aligned} 13x_1 + 19x_2 &\le 2400 && \text{machine time} \\ 20x_1 + 29x_2 &\le 2100 && \text{craftsman time} \\ x_1 &\ge 10 && \text{contract} \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}$$

9. Let x_1 be the number of units of P_1 produced in the current week and x_2 be the number of units of P_2 produced in the current week, then

$$z = (x_1 + 30 - 75) + (x_2 + 90 - 95) = x_1 + x_2 - 50 \to \max$$

$$\begin{aligned} 50x_1 + 24x_2 &\leq 2400 && \text{machine } A \text{ time} \\ 30x_1 + 33x_2 &\leq 2100 && \text{machine } B \text{ time} \\ x_1 &\geq 75(\text{demand}) - 30(\text{initial stock}) = 45 \\ x_2 &\geq 95(\text{demand}) - 90(\text{initial stock}) = 5 \end{aligned}$$

The aim is to maximize the number of units left in stock at the end of the week.

10. Let x_1 be the first and x_2 be the second number, then

$$z = x_1 + x_2 \to \max$$

$$\begin{aligned} x_1 - x_2 &\geq 4 \\ 3x_1 + x_2 &\leq 9 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$