ISS0031 Modeling and Identification

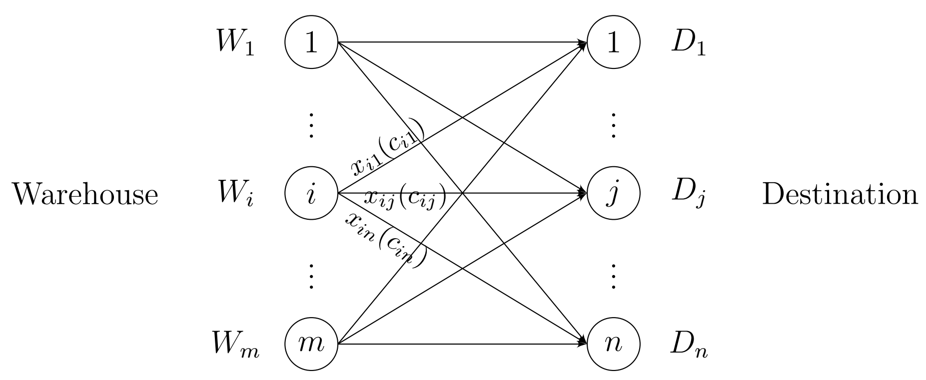
Lecture 10a

Introduction

One of the most important and successful applications of quantitative analysis to solving business problems has been in the physical distribution of products, commonly referred to as **transportation problems**. Basically, the purpose is to minimize the cost of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity.

Transportation problem

Transportation problems deal with the determination of a minimum-cost plan for transporting a commodity from a number of sources to a number of destinations. To be more specific, let there be m warehouses $W_1, \dots, W_m$ that have the commodity and n destinations (or consumers) $D_1, \dots, D_n$ that demand the commodity. At the ith warehouse, $i = 1, 2, \dots, m$, there are a_i units of the commodity available. The demand at the jth destination, $j = 1, 2, \dots, n$, is denoted by b_j. The cost of transporting one unit of the commodity from the ith warehouse to the jth destination (route W_iD_j) is c_{ij}. Let x_{ij} be the numbers of the commodity that are being transported from the ith warehouse to the jth destination. Our problem is to determine those x_{ij} that will minimize the overall transportation cost. An optimal solution x_{ij} to the problem is called a **transportation plan**.



Summarize some facts. The cost of transportation from W_i $(i = 1, 2, \dots, m)$ to D_j $(j = 1, 2, \dots, n)$ will be equal to

$$z = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \to \min . \tag{1}$$

Note that it is not possible to export from the warehouse W_i more than a_i:

$$\begin{aligned} x_{11} + x_{12} + \cdots + x_{1n} &\leq a_1 \\ x_{21} + x_{22} + \cdots + x_{2n} &\leq a_2 \\ &\vdots \\ x_{m1} + x_{m2} + \cdots + x_{mn} &\leq a_m \end{aligned}$$

or shortly

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots, m. \tag{2}$$

Note that the consumer at destination D_j needs b_j commodity or more:

$$\begin{aligned} x_{11} + x_{21} + \cdots + x_{m1} &\geq b_1 \\ x_{12} + x_{22} + \cdots + x_{m2} &\geq b_2 \\ &\vdots \\ x_{1n} + x_{2n} + \cdots + x_{mn} &\geq b_n \end{aligned}$$

or shortly

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n. \tag{3}$$

With the help of the above information we can construct the following table

Warehouse	Destination				Reserve
	D_1	D_2	$\cdots$	D_n	
W_1	c_{11}	c_{12}	$\cdots$	c_{1n}	a_1
W_2	c_{21}	c_{22}	$\cdots$	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	a_m
Requirement	b_1	b_2	$\cdots$	b_n	

Denote by $b = b_1 + b_2 + \cdots + b_n$ the total requirement of commodities and by $a = a_1 + a_2 + \cdots + a_m$ the total amount of available commodities.

Theorem 1. *Transportation problem* (1)-(3) *is solvable if and only if* $b \leq a$.

Proof: *Sufficiency*: Let transportation problem (1)-(3) has a solution x_{ij}, i.e. constraints (2)-(3) are satisfied and show that $b \leq a$ holds.

Let $l = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$ and bound it in two ways:

$$l = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) \leq \sum_{i=1}^{m} a_i = a$$

and

$$l = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right) \geq \sum_{j=1}^{n} b_j = b.$$

It is easy to observe that $b \leq l \leq a$.

Necessity: Suppose that $b \leq a$ holds, i.e. $\frac{b}{a} \leq 1$ and show that there exists some feasible solution. Consider point $(x_{11}^0, x_{12}^0, \ldots, x_{mn}^0)$, where $x_{ij}^0 = \frac{a_i b_j}{a}$. Next, substitute x_{ij}^0 into inequalities (2)

$$\sum_{j=1}^{n} x_{ij}^0 = \sum_{j=1}^{n} \frac{a_i b_j}{a} = \frac{a_i}{a}\sum_{j=1}^{n} b_j = \frac{a_i b}{a} = a_i \frac{b}{a} \leq a_i,$$

meaning that for each i the corresponding inequality in (2) holds. Analogously, substitute x_{ij}^0 into inequalities (3)

$$\sum_{i=1}^{m} x_{ij}^0 = \sum_{i=1}^{m} \frac{a_i b_j}{a} = \frac{b_j}{a}\sum_{i=1}^{m} a_i = \frac{b_j a}{a} = b_j,$$

meaning that (3) hold. As a result, we get that the point $(x_{11}^0, x_{12}^0, \ldots, x_{mn}^0)$ is a feasible solution, or alternatively that the set of feasible solutions is non-empty. Note that if the set of feasible solutions is bounded and the objective function is bounded from below, then there exists an optimal solution. The latter comes from two observations. First, since $x_{ij} \geq 0$ and $\sum_{j=1}^{n} x_{ij} \leq a_i$ by (2), $x_{ij} \leq a_i$ for each i, j. Therefore, the set of feasible solutions is bounded. Second, $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \geq 0$. Thus, the objective function is bounded from below. ∎

Remark 1. *If $b > a$, then the transportation problem is not solvable. In this case one has to solve non-mathematical problem either to increase reserve of commodity, or to decrease requirements.*

Definition 1. *If $a = b$, then transportation problem* (1)-(3) *is called* ***balanced****.*

Remark 2. *For the balanced transportation problem constraints* (2) *and* (3) *are of the form $\sum_{j=1}^{n} x_{ij} = a_i$ and $\sum_{i=1}^{m} x_{ij} = b_j$, respectively.*

Remark 3. *If the transportation problem is not in the balanced form, i.e. $b < a$, then one may introduce a fictive destination D_f with requirement $b_f = a - b$, getting the problem in the balanced form.*

Example 1: Consider the transportation problem given by the following table

Warehouse	Destination				Reserve
	D_1	D_2	D_3	D_4	
W_1	3	1	0	4	15
W_2	1	2	5	2	20
W_3	3	8	11	0	25
Requirement	10	10	15	20	

Let us introduce the fictive destination D_f in which a consumer needs 5 units of goods. Since this destination is fictive, the transportation costs can be taken equal to zero.

Warehouse	Destination					Reserve
	D_1	D_2	D_3	D_4	D_f	
W_1	3	1	0	4	0	15
W_2	1	2	5	2	0	20
W_3	3	8	11	0	0	25
Requirement	10	10	15	20	5	

Basic theorems

In the case of an unbalanced model, i.e. the total demand is not equal to the total supply, we can always add dummy source or dummy destination to complement the difference. In the following, we only consider balanced transportation models. They can be written as the following linear programming problem

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

subject to constraints

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i \quad 1 \le i \le m, \\ \sum_{i=1}^{m} x_{ij} &= b_j \quad 1 \le j \le n, \\ x_{ij} &\ge 0 \quad 1 \le i \le m, 1 \le j \le n, \end{aligned} \tag{4}$$

where $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Using the vector notations

$$\begin{aligned} x &= \left[x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}\right]^{\mathrm{T}}, \\ c &= \left[c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{2n}, \ldots, c_{m1}, \ldots, c_{mn}\right]^{\mathrm{T}}, \\ b &= \left[a_1, \ldots, a_m, b_1, \ldots, b_n\right]^{\mathrm{T}}, \end{aligned}$$

the transportation model can be rewritten using matrix notation

$$\begin{aligned} z &= c^{\mathrm{T}} x \to \min \\ Ax &= b \\ x &\ge 0 \end{aligned}$$

Theorem 2. *The rank of the matrix A is equal to $n + m - 1$.*

Proof: Let us find the rank of the matrix A using elementary operations. Present matrix A in the table form

	x_{11}	x_{12}	$\dots$	x_{1n}	x_{21}	x_{22}	$\dots$	x_{2n}	$\dots$	x_{m1}	x_{m2}	$\dots$	x_{mn}	b
(1)	1	1	$\cdots$	1	0	0	$\cdots$	0	$\cdots$	0	0	$\cdots$	0	a_1
(2)	0	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	0	0	$\cdots$	0	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(m)	0	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	1	1	$\cdots$	1	a_m
$(m+1)$	1	0	$\cdots$	0	1	0	$\cdots$	0	$\cdots$	1	0	$\cdots$	0	b_1
$(m+2)$	0	1	$\cdots$	0	0	1	$\cdots$	0	$\cdots$	0	1	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(m+n)$	0	0	$\cdots$	1	0	0	$\cdots$	1	$\cdots$	0	0	$\cdots$	1	b_n

Next, apply the operations $R_1 - R_{m+1} - \cdots - R_{m+n} + R_2 + \cdots + R_m$ and $R_{m+1} - R_2 - \cdots - R_m$. The new table becomes

	x_{11}	x_{12}	$\dots$	x_{1n}	x_{21}	x_{22}	$\dots$	x_{2n}	$\dots$	x_{m1}	x_{m2}	$\dots$	x_{mn}	b
(1)	0	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	$\cdots$	0	$\ast_1$
(2)	0	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	0	0	$\cdots$	0	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(m)	0	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	1	1	$\cdots$	1	a_m
$(m+1)$	1	0	$\cdots$	0	0	-1	$\cdots$	-1	$\cdots$	0	-1	$\cdots$	-1	$\ast_2$
$(m+2)$	0	1	$\cdots$	0	0	1	$\cdots$	0	$\cdots$	0	1	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(m+n)$	0	0	$\cdots$	1	0	0	$\cdots$	1	$\cdots$	0	0	$\cdots$	1	b_n

where $\ast_1 = a_1 + \cdots + a_m - b_1 - \cdots - b_m = a - b = 0$ and $\ast_2 = b_1 - a_2 - \cdots - a_m$. Finally, we get the matrix

$$M = \begin{pmatrix} \theta_{mn} & E_{m,m-1} \\ E_{nn} & \theta_{n,m-1} \end{pmatrix}$$

Now, it is easy to see that $|M| \neq 0$ and $\text{rank}|M| = m + n - 1$. ■

Dual problem of the balanced transportation problem: Notice that there are $m+n$ variables $y_1, y_2, \dots, y_{m+n}$. Denote them by $y_1 = u_1, y_2 = u_2, \dots, y_m = u_m, y_{m+1} = v_1, y_{m+2} = v_2, \dots, y_{m+n} = v_n$. Then the dual transportation problem can be written as

$$w = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \to \max$$

subject to constraints

$$u_i + v_j \le c_{ij} \quad 1 \le i \le m, 1 \le j \le n.$$

Remark 4. *By Theorem 2 the rank of the matrix A for both primal and dual problems is equal to* $m + n - 1$.

It is interesting to note that

Primal problem: The total number of variables is mn. There are $m+n-1$ basic variables and $mn - (m+n-1)$ free variables.

Dual problem: The total number of variables is $m+n$. There are $m+n-1$ basic variables and $m+n-(m+n-1) = 1$ free variable.

Theorem 3. *The feasible solutions* $(x_{11}^*, x_{12}^*, \ldots, x_{1n}^*, \ldots, x_{m1}^*, x_{m2}^*, \ldots, x_{mn}^*)$ *and* $(u_1^*, u_2^*, \ldots, u_m^*, v_1^*, v_2^*, \ldots, v_n^*)$ *of the balanced primal and dual transportation problems, respectively, are optimal solutions if and only if*

$$(c_{ij} - (u_i^* + v_j^*))x_{ij}^* = 0 \tag{5}$$

for each $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$.

Proof: Note that $(x_{11}^*, x_{12}^*, \ldots, x_{1n}^*, \ldots, x_{m1}^*, x_{m2}^*, \ldots, x_{mn}^*)$ is a feasible solution of the primal transportation problem, meaning that $\sum_{j=1}^n x_{ij}^* = a_i$ and $\sum_{i=1}^m x_{ij}^* = b_j$; $(u_1^*, u_2^*, \ldots, u_m^*, v_1^*, v_2^*, \ldots, v_n^*)$ is a feasible solution of the dual transportation problem, meaning that $u_i^* + v_j^* \le c_{ij}$.

Sufficiency: Suppose that $(x_{11}^*, \ldots, x_{1n}^*, \ldots, x_{m1}^*, \ldots, x_{mn}^*)$ and $(u_1^*, \ldots, u_m^*, v_1^*, \ldots, v_n^*)$ are optimal solutions

$$\begin{aligned} z_{\min} &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}^*, \\ w_{\max} &= \sum_{i=1}^m a_i u_i^* + \sum_{j=1}^n b_j v_j^* = \\ & \sum_{i=1}^m \left(\sum_{j=1}^n x_{ij}^* \right) u_i^* + \sum_{j=1}^n \left(\sum_{i=1}^m x_{ij}^* \right) v_j^* = \\ & \sum_{i=1}^m \sum_{j=1}^n u_i^* x_{ij}^* + \sum_{i=1}^m \sum_{j=1}^n v_j^* x_{ij}^* = \\ & \sum_{i=1}^m \sum_{j=1}^n \left(u_i^* x_{ij}^* + v_j^* x_{ij}^* \right) = \sum_{i=1}^m \sum_{j=1}^n \left(u_i^* + v_j^* \right) x_{ij}^*. \end{aligned}$$

By Theorem 2 from Lecture 5 we know that $z_{\min} = w_{\max}$

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}^* - \sum_{i=1}^m \sum_{j=1}^n \left(u_i^* + v_j^* \right) x_{ij}^* = 0$$

or

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}x_{ij}^* - \left(u_i^* + v_j^*\right)\right)x_{ij}^* = 0.$$

Note that $c_{ij}x_{ij}^* - (u_i^* + v_j^*) \geq 0$ and $x_{ij}^* \geq 0$. The sum of non-negative terms is equal to zero if and only if each term is equal to zero meaning that (5) holds.

Necessity: Let us sum equalities (5) over all the indices i, j. Next, we can make arguments, presented above, in the reverse order. ■

Problems

10a.1: Prove the following theorem

Theorem 4. *Every minor of A can only have one of the values 1, -1 or 0. More precisely, given any A_k, a k-by-k submatrix of A, we have $\det A_k = \pm 1$ or 0.*