ISS0031 Modeling and Identification

Lecture 10b

Introduction

Recall that the transportation problem is to minimize the function

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

subject to constraints

$$\sum_{j=1}^{n} x_{ij} \leq a_i$$
$$\sum_{i=1}^{m} x_{ij} \geq b_j$$
$$x_{ij} \geq 0$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

In previous lessons we have considered a number of specific linear programming problems. Transportation problems are also linear programming problems and can be solved by the Simplex method. Notice that there are mn variables but only $m + n$ equations. To initiate the Simplex method, we have to add $m + n$ more artificial variables and solving the problem by the Simplex method seems to be a very tedious task even for moderate values of m and n. Therefore, because of practical significance and the special structure of the transportation problem we can solve it with a faster, more economical algorithm than simplex.

Solution of the transportation problem: method of multipliers

Preliminary step:
Using Theorem 1 from Lecture 6, we have to check whether the stated problem is solvable or not. In the case of an unbalanced model, i.e. the total demand is not equal to the total supply, we can always add dummy source or dummy destination to complement the difference.

The starting basic feasible solution:
First of all, we have to find a starting basic feasible solution to the transportation problem. There are many methods for finding such a starting BFS. The easier ones are the *northwest-corner method*, the *column minima method* and *the row-minima method*. In the following, we explain the **northwest-corner method**. In this method we distribute the available units in rows and column in such a way that the sum will remain the same. We have to follow the steps given below.

a) Select the north west (upper left-hand) corner (cell) of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e. $\min(a_1, b_1)$.

b) Adjust the supply and demand numbers in the respective rows and columns.

c) If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.

d) If the supply for the first row is exhausted, then move down to the first cell in the second row.

e) If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.

f) Continue the process until all supply and demand values are exhausted.

Example 1: There are 3 warehouses W_i for $i = 1, \dots, 3$ with commodity of the same type in amount of $a_1 = 8$, $a_2 = 10$, $a_3 = 20$ units, respectively, and there are 4 destinations (consumers) D_j for $j = 1, \dots, 4$ who want to receive at least $b_1 = 6$, $b_2 = 8$, $b_3 = 9$, $b_4 = 15$ units of the commodity, respectively. The cost of transporting one unit of the commodity from warehouse W_i to consumer D_j together with available information are summarized in the following table:

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2	3	5	1	8
W_2	7	3	4	6	10
W_3	4	1	7	2	20
Requirement	6	8	9	15	38

One may see that the problem is balanced, since $\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 38$.

Start allocations from north-west corner, i.e. from (1,1) position. Here $\min(a_1, b_1) = \min(8, 6) = 6$ units. Therefore, the maximum possible units that can be allocated to this position is 6. This completes the allocation in the first column and cross the other positions, i.e. (2,1) and (3,1) in the column, see the table below. Note that in the following tables the transportation costs c_{ij} are placed in the left-upper corner of the cell.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3}	^{5}	^{1}	$8 - 6 = 2$
W_2	^{7} $\times$	^{3}	^{4}	^{6}	10
W_3	^{4} $\times$	^{1}	^{7}	^{2}	20
Requirement	$6 - 6 = 0$	8	9	15	38

After completion of the previous step, come across the position (1,2). Here $\min(8-6, 8) = 2$ units can be allocated to this position. This completes the allocations in the first row and cross the other positions, i.e. (1,3) and (1,4) in this row.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3} **2**	^{5} ×	^{1} ×	$2-2=0$
W_2	^{7} ×	^{3}	^{4}	^{6}	10
W_3	^{4} ×	^{1}	^{7}	^{2}	20
Requirement	0	$8-2=6$	9	15	30

Now, we go to the second row, here the position (2,1) is already been struck off, so consider the position (2,2). Here $\min(10, 8-2) = 6$ units can be allocated to this position. This completes the allocations in second column so strike off the position (3,2).

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3} **2**	^{5} ×	^{1} ×	0
W_2	^{7} ×	^{3} **6**	^{4}	^{6}	$10-6=4$
W_3	^{4} ×	^{1} ×	^{7}	^{2}	20
Requirement	0	0	9	15	24

Again consider the position (2,3). Here, $\min(10-6, 9) = 4$ units can be allocated to this position. This completes the allocations in second row so struck off the position (2,4).

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3} **2**	^{5} ×	^{1} ×	0
W_2	^{7} ×	^{3} **6**	^{4} **4**	^{6} ×	0
W_3	^{4} ×	^{1} ×	^{7}	^{2}	20
Requirement	0	0	$9-4=5$	15	20

In the third row, positions (3,1) and (3,2) are already been struck off so consider the position (3,3) and allocate it the maximum possible units, i.e. $\min(20, 9-4) = 5$ units. Finally, allocate the remaining units to the position (3,4), i.e. 15 units to this position. Keeping in mind all the allocations done in the above method complete the table as follows.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} **6**	^{3} **2**	^{5} $\times$	^{1} $\times$	8
W_2	^{7} $\times$	^{3} **6**	^{4} **4**	^{6} $\times$	10
W_3	^{4} $\times$	^{1} $\times$	^{7} **5**	^{2} **15**	20
Requirement	6	8	9	15	38

From the above table we can see that the starting basic feasible solution is $x_{11} = 6$, $x_{12} = 2$, $x_{21} = 6$, $x_{22} = 4$, $x_{32} = 5$, $x_{34} = 15$, and the other variables in the table are $x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$. Therefore, the cost of transportation can be calculated as

$$z = c_{11}x_{11} + c_{12}x_{12} + c_{22}x_{22} + c_{23}x_{23} + c_{33}x_{33} + c_{34}x_{34} =$$
$$= 12 + 6 + 18 + 16 + 35 + 30 = 117.$$

Solution algorithm:
Step 0. Assume that preliminary step is accomplished and starting feasible solution is found.

Step 1. Find the multipliers u_i, $i = 1, \ldots, m$ and v_j, $j = 1, \ldots, n$ from the relations $u_i + v_j = c_{ij}$ for all (i, j)-cells containing basic variables. Since there are $m + n - 1$ basic variables, we get the same number of equations. However, there are $m + n$ unknown variables u_i and v_j. Therefore, one of the variables may be fixed, say equal to zero (for example $v_1 = 0$), and the equations may be used to solve for the other variables. Some of the u_i or v_j may turn out to be negative, but this does not matter. Find the indirect transportation costs as $\hat{c}_{ij} = u_i + v_j$ for (i, j)-cells containing non-basic variables.

Step 2. Calculate[1] $\varphi = \max(\hat{c}_{ij} - c_{ij})$. Check the **optimality criteria**, which is $\varphi = 0$. If it is satisfied, then the obtained transportation plan is optimal. Otherwise, the plan can be improved. It means that we have to redistribute some amount of the commodity, say θ, which has to be put to the cell for which the difference $\hat{c}_{ij} - c_{ij}$ is maximal. However, if we add θ to that cell, we must subtract and add θ to other cells containing basic variables to keep the constraints (requirements vs. reserve) satisfied. We choose θ as large as possible, bearing in mind that negative shipments are not allowed. It means that at least one of the basic variables is put, or remains at, 0.

Step 3. Repeat Steps 1 and 2 until the optimality criteria is satisfied.

Remark 1. *Given the optimal solution x^*. If there exist a cell in which $\hat{c}_{ij} - c_{ij} = 0$ and this is not a cell containing basic variable, then the optimal solution is not unique and there is an alternative optimal plan, which can be found by putting θ to this cell and repeating Steps* 1-3.

Example 2: (Continuation of Example 1). Recall the transportation table obtained in Example 1.

[1]Note that for the cells containing basic variables the difference $\hat{c}_{ij} - c_{ij}$ is always zero.

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	^{2} 6	^{3} 2	^{5}	^{1}	8
W_2	^{7}	^{3} 6	^{4} 4	^{6}	10
W_3	^{4}	^{1}	^{7} 5	^{2} 15	20
Requirement	6	8	9	15	38

According to the algorithm presented above we can construct the following system of equations and solve it for u_i, $i = 1, \dots, 3$ and v_j, $j = 1, \dots, 4$

$$\begin{cases} u_1 + v_1 = 2 \\ u_1 + v_2 = 3 \\ u_2 + v_2 = 3 \\ u_2 + v_3 = 4 \\ u_3 + v_3 = 7 \\ u_3 + v_4 = 2 \end{cases}$$

By assigning $v_1 = 0$, we get $u_1 = 2, u_2 = 2, u_3 = 5$ and $v_1 = 0, v_2 = 1, v_3 = 2, v_4 = -3$. Now, we can add additional column and row and rewrite the transportation table as follows:

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	^{2} 6	^{3} 2	^{5}	^{1}	8	2
W_2	^{7}	^{3} 6	^{4} 4	^{6}	10	2
W_3	^{4}	^{1}	^{7} 5	^{2} 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Next, we calculate the indirect transportation costs as $\hat{c}_{ij} = u_i + v_j$ for (i, j)-cells containing non-basic variables. Note that in the following table $\hat{c}_{ij}$ is placed in the left-down corner of the cell.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 2	5 4	1 -1	8	2
W_2	7 2	3 6	4 4	6 -1	10	2
W_3	4 5	1 6	7 5	2 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

After that, we calculate the difference between the indirect and actual transportation costs as $\hat{c}_{ij} - c_{ij}$. Note that in the following table the difference is placed in the right-upper corner of the cell.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 2	5 -1 4	1 -2 -1	8	2
W_2	7 -5 2	3 6	4 4	6 -7 -1	10	2
W_3	4 1 5	1 5 6	7 5	2 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Now, we can easily see that $\varphi = \max(\hat{c}_{32} - c_{32}) = 5$. Since the optimality condition is not satisfied, the transportation plan can be improved. It means that we have to redistribute some amount of the commodity. For that purpose we add θ to the cell (3,2). Since we added θ to that cell, we must subtract it from cells (2,2) and (3,3), respectively. Finally, we have to add θ to the cell (2,3) to keep the constraints satisfied.

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 2	5 -1 4	1 -2 -1	8	2
W_2	7 -5 2	3 $6-\theta$	4 $4+\theta$	6 -7 -1	10	2
W_3	4 1 5	1 θ 5 6	7 $5-\theta$	2 15	20	5
Requirement	6	8	9	15	38	
v_j	0	1	2	-3		

Doing this way we can see that $\theta = \max(6-\theta, 4+\theta, 5-\theta) = 5$. After modifying the transportation plan the new table becomes:

Warehouse	Destinations				Reserve
	D_1	D_2	D_3	D_4	
W_1	2 6	3 2	5	1	8
W_2	7	3 1	4 9	6	10
W_3	4	1 5	7	2 15	20
Requirement	6	8	9	15	38

From the above table we can see that the cost of transportation can be calculated as

$$z = c_{11}x_{11} + c_{12}x_{12} + c_{22}x_{22} + c_{23}x_{23} + c_{32}x_{32} + c_{34}x_{34} = \\ = 12 + 6 + 3 + 36 + 5 + 30 = 107,$$

which decreased after modifying the plan. However, it is not hard to check that this cost is still not optimal. Therefore, repeating Steps 1-3 of the algorithm, we get the following tables:

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 $2-\theta$	5 -1 4	1 θ 3 4	8	2
W_2	7 -5 2	3 1	4 9	6 -2 4	10	2
W_3	4 -4 0	1 $5+\theta$	7 -5 2	2 $15-\theta$	20	0
Requirement	6	8	9	15	38	
v_j	0	1	2	2		

for which $\theta = 2$, and

Warehouse	Destinations				Reserve	u_i
	D_1	D_2	D_3	D_4		
W_1	2 6	3 -3 0	5 -4 1	1 2	8	2
W_2	7 -2 5	3 1	4 9	6 -2 4	10	5
W_3	4 -1 3	1 7	7 -5 2	2 13	20	3
Requirement	6	8	9	15	38	
v_j	0	-2	-1	-1		

for which the optimality criteria is satisfied and the minimal transportation cost is therefore

$$z = c_{11}x_{11} + c_{14}x_{14} + c_{22}x_{22} + c_{23}x_{23} + c_{32}x_{32} + c_{34}x_{34} = \\ = 12 + 2 + 3 + 36 + 7 + 26 = 86.$$

Exercises

Example 3: There are 3 warehouses W_i for $i = 1, \dots, 3$ with commodity of the same type in amount of $a_1 = 28$, $a_2 = 56$, $a_3 = 56$ units, respectively, and there are 4 consumers D_j for $j = 1, \dots, 4$ who want to receive at least $b_1 = 49$, $b_2 = 14$, $b_3 = 42$, $b_4 = 42$ units of the commodity, respectively. The cost of transporting one unit of the commodity from warehouse W_i to consumer D_j together with available information are summarized in the following table:

Warehouse	Destinations (consumers)				Reserve
	D_1	D_2	D_3	D_4	
W_1	5	4	2	1	28
W_2	4	3	2	3	56
W_3	4	3	3	6	56
Requirement	49	14	42	42	

Meet the consumer requirements at minimum transportation cost.

Example 4:

	D_1	D_2	D_3	D_4	Reserve
W_1	5	1	2	1	10
W_2	2	0	3	2	20
W_3	4	1	1	2	30
Requirement	25	15	10	10	

Example 5:

	D_1	D_2	D_3	D_4	D_5	Reserve
W_1	16	30	17	10	16	4
W_2	30	27	26	9	23	6
W_3	13	4	22	3	1	10
W_4	3	1	5	4	24	10
Requirement	7	7	7	7	2	

Example 6:

	D_1	D_2	D_3	D_4	D_5	Reserve
W_1	5	15	3	6	10	9
W_2	23	8	13	27	12	11
W_3	30	1	5	24	25	14
W_4	8	26	7	28	9	16
Requirement	8	9	13	8	12	

Problems

Solve the following transportation problems.

10b.1:

	D_1	D_2	D_3	D_4	D_5	Reserve
W_1	2	9	4	10	6	280
W_2	7	3	0	5	0	320
W_3	5	2	1	7	8	240
W_4	11	6	2	3	4	160
Requirement	130	250	170	100	300	

10b.2:

	D_1	D_2	D_3	D_4	D_5	Reserve
W_1	5	8	7	10	3	1000
W_2	4	2	2	5	6	2250
W_3	7	3	5	9	2	1250
Requirement	500	625	1625	1250	1000	

10b.3:

	D_1	D_2	D_3	D_4	D_5	Reserve
W_1	5	8	7	10	3	200
W_2	4	2	2	5	6	450
W_3	7	3	5	9	2	250
Requirement	100	125	325	250	200	

10b.4:

	D_1	D_2	D_3	D_4	Reserve
W_1	18	13	9	13	284
W_2	7	5	8	19	566
W_3	9	2	4	17	170
W_4	14	6	1	6	280
Requirement	145	625	200	300	

10b.5:

	D_1	D_2	D_3	D_4	Reserve
W_1	5	8	11	6	3120
W_2	8	23	4	7	2880
W_3	7	19	3	5	1290
W_4	11	8	5	4	510
Requirement	2580	690	1500	3030	

Answers to problems

1. $z_{\min} = 1590$,

$$x_{ij} = \begin{pmatrix} 130 & 0 & 100 & 0 & 0 \\ 0 & 10 & 10 & 0 & 300 \\ 0 & 240 & 0 & 0 & 0 \\ 0 & 0 & 60 & 100 & 0 \end{pmatrix}, \qquad i = 1, \ldots, 4, j = 1, \ldots, 5.$$

2. $z_{\min} = 14250$,

$$x_{ij} = \begin{pmatrix} 500 & 0 & 0 & 0 & 500 \\ 0 & 0 & 1500 & 750 & 0 \\ 0 & 625 & 125 & 0 & 500 \end{pmatrix}, \qquad i = 1, \ldots, 3, j = 1, \ldots, 5.$$

3. $z_{\min} = 2850$,

$$x_{ij} = \begin{pmatrix} 100 & 0 & 0 & 0 & 100 \\ 0 & 0 & 300 & 150 & 0 \\ 0 & 125 & 25 & 0 & 100 \end{pmatrix}, \qquad i = 1, \ldots, 3, j = 1, \ldots, 5.$$

4. $z_{\min} = 8642$, $x_{ij} = \alpha x_1 + (1-\alpha)x_2$, $0 \le \alpha \le 1$, $i, j = 1, \ldots, 4$,

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 220 \\ 145 & 421 & 0 & 0 \\ 0 & 170 & 0 & 0 \\ 0 & 34 & 200 & 80 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 34 & 0 & 220 \\ 145 & 421 & 0 & 0 \\ 0 & 170 & 0 & 0 \\ 0 & 0 & 200 & 80 \end{pmatrix}.$$

5. $z_{\min} = 41970$, $x_{ij} = \alpha x_1 + (1-\alpha)x_2$, $0 \le \alpha \le 1$, $i, j = 1, \ldots, 4$,

$$x_1 = \begin{pmatrix} 2580 & 540 & 0 & 0 \\ 0 & 0 & 1500 & 1380 \\ 0 & 0 & 0 & 1290 \\ 0 & 150 & 0 & 360 \end{pmatrix}, x_2 = \begin{pmatrix} 2430 & 690 & 0 & 0 \\ 150 & 0 & 1500 & 1230 \\ 0 & 0 & 0 & 1290 \\ 0 & 0 & 0 & 510 \end{pmatrix}.$$