ISS0031 Modeling and Identification

Lecture 2

System of linear equations: solution set

Let us consider a simple problem in two variables x_1 and x_2. Find x_1 and x_2 which satisfy the following equations

$$\begin{aligned} x_1 + x_2 &= 5 \\ 2x_1 + 3x_2 &= 7 \end{aligned}$$

Solving these equations, we get $x_1 = 8$ and $x_2 = -3$. However, one may ask: what happens when the number of equations and variables will be greater or less?

Let n be the number of unknowns and m be the number of equations. A linear system may behave in any one of three possible ways:

1. If $m < n$, then a system has infinitely many solutions (underdetermined system).
2. If $m = n$, then a system has a single unique solution.
3. If $m > n$, then a system has no solutions (overdetermined system).

Here we may ask another question: if relations are in the form of inequalities, can we find a solution for such a system? Whenever the analysis of a problem leads to minimizing or maximizing a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution** of the linear programming problem.

Linear programming problem: a geometric approach

To solve a linear programming problem involving two variables by the graphical method, use the following steps

1. Formulate the linear programming problem.
2. Graph the constraints inequalities.
3. Identify the feasible region which satisfies all the constraints simultaneously. For "less than or equal to" constraints the region is generally below the lines and "for greater than or equal to" constraints, the region is above the lines.
4. Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.

5. Evaluate the objective function at each of the vertex (corner point).
6. Identify the optimum value of the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, if an optimal solution exists, then it will occur at a vertex.)

Optimal solution of a linear programming problem:

- If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions.
- If the problem has more than one solution, then at least one of them must occur at a vertex of the set of feasible solutions.
- If none of the feasible solutions maximizes (or minimizes) the objective function, or if there are no feasible solutions, then the linear programming problem has no solutions.

Next, we illustrate the discussed approach by the number of numerical examples.

Example 1: Consider the following linear programming problem

$$z = x_1 + 2x_2 \to \max(\min)$$
$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0, x_2 \geq 0.$$

Draw the graphs of these inequalities, which is as follows.

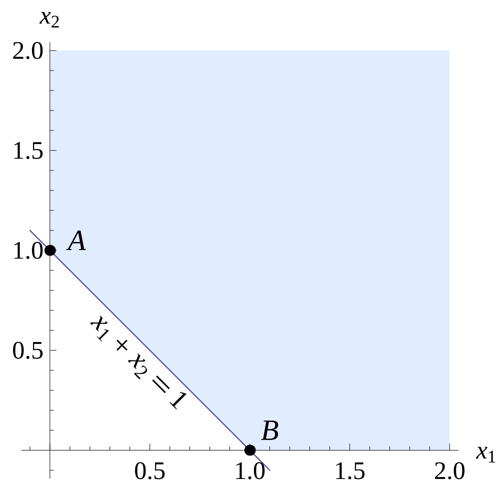

Figure 1: Feasible region (light blue)

The region satisfied by $x_1 \geq 0$ and $x_2 \geq 0$ is the first quadrant and the region satisfied by the line $x_1 + x_2 \geq 0$ along with $x_1 \geq, x_2 \geq 0$ will be on that side of the line $x_1 + x_2 = 1$ in which the origin is not located. Every point in the region satisfies all the mathematical inequalities and hence the feasible solution. Now, we have to find optimal solution. The vertices of the feasible region are $A(1, 0)$ and $B(0, 1)$. The value of z at A is 1. The value of z at B is 2.

- **Minimum:** We can immediately conclude that the value of z is minimum at $A(1,0)$, i.e. $z_{\min}(1,0) = 1$.

- **Maximum:** If we take the value of z at any other point from the feasible region, then we notice that every time we can find another point which gives the larger value than the previous one. It is due to the fact that the region, determined by the constraints, is unbounded. Hence, there is no feasible point that will make z largest. Therefore, we conclude that this linear programming problem has no solution in case of maximizing the objective function.

Example 2: Consider the following linear programming problem

$$z = x_1 + 2x_2 \to \min$$
$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0, x_2 \geq 0$$

Draw the graphs of these inequalities as follows.

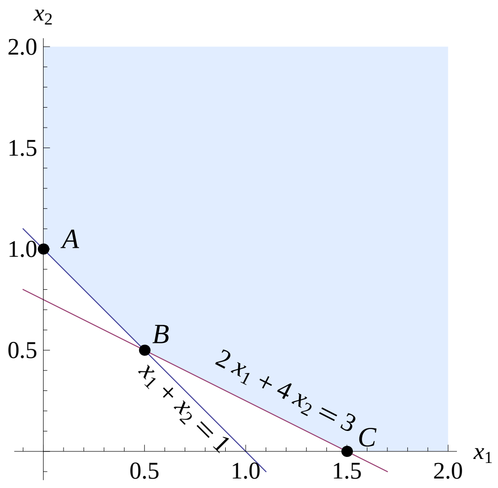

Figure 2: Feasible region (light blue)

The shaded region is the feasible region. Hence, the shaded region is our feasible solution because every point in this region satisfies all the constraints. Now, we have to find the optimal solution. First, we check the vertices of the region: $z(A) = 2$, $z(B) = 1.5$, and $z(C) = 1.5$. In this case, we can conclude that the objective function has a minimum value not only at the vertices B and C, it also has a minimum value at any point on the line segment connecting these two vertices, i.e $x_1 = \frac{3-2\alpha}{2}, x_2 = 0.5\alpha, 0 \leq \alpha \leq 1$. Of course, the reason that any feasible point (between B and C) on $2x_1 + 4x_2 = 3$ minimizes the objective function $z = x_1 + 2x_2$ is that the two lines are parallel (both have slope -0.5). Therefore, this linear programming problem has infinitely many solutions and two of them occur at the vertices.

Example 3: Consider the following linear programming problem

$$
\begin{aligned}
z = 2x_1 + x_2 &\to \max(\min) \\
x_1 + 2x_2 &\leq 2 \\
x_1 + x_2 &\leq 1.5 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

Draw the graphs of these inequalities as follows.

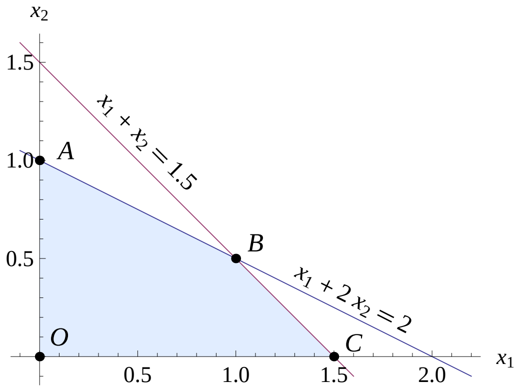

Figure 3: Feasible region (light blue)

The shaded region $OABC$ is the feasible region. Every point in the region satisfies all the mathematical inequations and hence the feasible solutions. Now, we have to find the optimal solution. We check the vertices of the region: $z(A) = 1$, $z(B) = 2.5$, $z(C) = 3$, and $z(O) = 0$. If we take any other value from the feasible region we see that still the maximum value is 3 obtained at the vertex $C(1.5, 0)$ of the feasible region. The same arguments hold for the minimum value $z_{\min}(0,0) = 0$.

Exercises

Solve graphically the linear programming problems:

Example 4:

$$
\begin{aligned}
z = 3x_1 + 2x_2 &\to \max \\
x_1 + 2x_2 &\leq 4 \\
x_1 - x_2 &\leq 1 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

Example 5:

$$\begin{aligned} z = 5x_1 + 7x_2 &\to \min \\ 2x_1 + 3x_2 &\geq 6 \\ 3x_1 - x_2 &\leq 15 \\ -x_1 + x_2 &\leq 4 \\ 2x_1 + 5x_2 &\leq 27 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 6:

$$\begin{aligned} z = 4x_1 + 3x_2 &\to \max(\min) \\ 2x_1 + 3x_2 &\geq 6 \\ 3x_1 - 2x_2 &\leq 9 \\ x_1 + 5x_2 &\leq 20 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 7:

$$\begin{aligned} z = 3x_1 + 4x_2 &\to \max \\ x_1 + x_2 &\leq 40 \\ x_1 + 2x_2 &\leq 60 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 8:

$$\begin{aligned} z = 3x_1 + 4x_2 &\to \max \\ x_1 + x_2 &\leq 40 \\ x_1 + 2x_2 &\leq 60 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 9:

$$\begin{aligned} z = 4x_1 - 3x_2 &\to \max(\min) \\ 3x_1 + x_2 &\leq 9 \\ -x_1 + x_2 &\leq 1 \\ x_1 + x_2 &\leq 6 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Example 10: A machine producing either product P_1 and P_2 can produce P_1 by using 2 units of chemicals and 1 unit of a compound and can produce P_1 by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1000 units of the compound are available. The profits available per unit of P_1 and P_2 are respectively 30 EUR and 20 EUR. Find the optimum allocation of units between P_1 and P_2 to maximize the total profit. Find the maximum profit.

Example 11: A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day. If each scientific calculator sold results in a 2EUR loss, but each graphing calculator produces a 5 EUR profit, how many of each type should be made daily to maximize net profits?

Example 12: A farmer has 10 acres to plant in wheat and rye. He has to plant at least 7 acres. However, he has only 1200 EUR to spend and each acre of wheat costs 200 EUR to plant and each acre of rye costs 100 EUR to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye. If the profit is 500 EUR per acre of wheat and 300 EUR per acre of rye how many acres of each should be planted to maximize profits?

Problems

Solve graphically the mathematical programming problems:

2.1:

$$\begin{gathered} z = 2x_1 - 2x_2 \to \max(\min) \\ x_1 - 3x_2 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 4x_1 + 9x_2 \leq 36 \\ x_1 \geq 0, x_2 \geq 0 \end{gathered}$$

2.2:

$$\begin{gathered} z = -2x_1 - 3x_2 \to \max(\min) \\ x_1 - 2x_2 \leq 0 \\ 6x_1 + 9x_2 \geq 27 \\ -x_1 + x_2 \leq 4 \\ x_2 \leq 5 \\ x_1 \geq 0, x_2 \geq 0 \end{gathered}$$

2.3:

$$\begin{gathered} z = x_1^2 + x_2^2 \to \max \\ 2x_1 + 3x_2 \leq 6 \\ x_1 \geq 0, x_2 \geq 0 \end{gathered}$$

2.4:

$$z = (x_1 - 1)^2 + (x_2 - 1)^2 \to \min$$
$$2x_1 + 3x_2 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

2.5:

$$z = 5x_1 - 4x_2 \to \max$$
$$x_1 + x_2 \geq 4$$
$$2x_1 + x_2 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

2.6:

$$z = 5x_1^2 - 8x_2 \to \max(\min)$$
$$3x_1 + x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

2.7:

$$z = x_1^2 x_2 \to \max(\min)$$
$$x_1^2 + x_2 \leq 12$$
$$x_1^2 + x_2^2 \leq 24$$
$$x_1 \geq 0, x_2 \geq 0$$

2.8:

$$z = 2x_1 + 4x_2 \to \max(\min)$$
$$-x_1 + 3x_2 \geq 0$$
$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \geq 2$$
$$x_1 \geq 0, x_2 \geq 0$$

2.9:

$$z = 6x_1 + 2x_2 \to \max(\min)$$
$$x_1 - x_2 \leq 2$$
$$3x_1 + x_2 \geq 3$$
$$-x_1 + x_2 \leq 5$$
$$x_1 < 6, x_2 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

2.10:

$$\begin{aligned} z = 14x_1 + 4x_2 &\to \max(\min) \\ 8x_1 + 9x_2 &\leq 72 \\ 7x_1 + 2x_2 &\geq 14 \\ -x_1 + x_2 &\leq 2 \\ 4x_1 - 7x_2 &\leq 14 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

2.11: A retired couple have up to 30000 EUR to invest in fixed-income securities. Their broker recommends investing in two bonds: one a AAA bond yielding 8%; the other a B+ bond paying 12%. After some consideration, the couple decide to invest at most 12 000 EUR in the B+-rated bond and at least 6000 EUR in the AAA bond. They also want the amount invested in the AAA bond to exceed or equal the amount invested in the B+ bond. What should the broker recommend if the couple want to maximize the return on their investment?

Answers to problems

1. $z_{\max}(9,0) = 18$ and $z_{\min}(0,0) = 0$.
2. $z_{\max}(0,3) = -9$ and $z_{\min}(0,5) = -35$.
3. $z_{\max}(3,0) = 9$.
4. $z_{\min}(1,1) = 0$.
5. $z_{\max}(2,2) = 2$.
6. $z_{\max}(\frac{10}{3}, 0) = \frac{500}{9}$ and $z_{\min}(0,6) = -48$.
7. $z_{\max}(\sqrt{8}, 4) = 8\sqrt{2}$ and $z_{\min}$ does not exist.
8. $z_{\max}(x_1, x_2) = 10$, $x_1 = 3 - 3\alpha, x_2 = 1 + \frac{3}{2}\alpha, 0 \leq \alpha \leq 1$ and $z_{\min}(\frac{3}{2}, \frac{1}{2}) = 8$.
9. $z_{\max}(6,6) = 48$ and $z_{\min}(x_1, x_2) = 6$, $x_1 = \alpha, x_2 = 3 - 3\alpha, 0 \leq \alpha \leq 1$.
10. $z_{\max}(\frac{315}{46}, \frac{44}{23}) = \frac{2293}{23}$ and $z_{\min}(x_1, x_2) = 28$, $x_1 = 2 - \frac{8}{9}\alpha, x_2 = \frac{28}{9}\alpha$.
11. The maximum return on investment is 2880 EUR, obtained by placing 18000 in the AAA bond and 12000 in the B+ bond, i.e. $z_{\max}(18000, 12000) = 2880$.