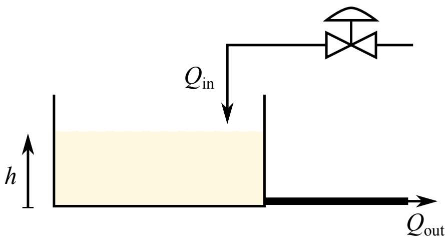
Process description: diff. equation, linearization

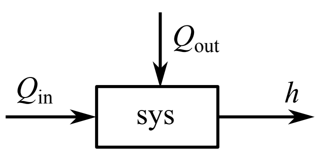


Aim: height control.
What is the dependance between flows and liquid level?
Mass conservation law

$m_{acc} = m_{in} - m_{out}$, if $\rho = \text{const}$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = Q_{in} - Q_{out}$$
$$V = A \cdot h$$

1 Differential equation



$$h = \frac{1}{A}\int (Q_{in} - Q_{out})\mathrm{d}t \tag{1}$$

What is the Q_{out}?
It depends on the pressure on the buttom

$$P = \frac{F}{A} = \frac{m \cdot g}{A} = \frac{\rho \cdot V \cdot g}{A} = \frac{\rho \cdot h \cdot A \cdot g}{A} = \rho h g$$

In that case

$$Q_{out} \equiv n^* \sqrt{P} \qquad \Rightarrow \qquad Q_{out} = n\sqrt{h},$$

where n^* - coeff. dependant on the properties of the liquid and valve.
Thus, flow rate through a restriction such as discharge valve is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{A}(Q_{in} - n\sqrt{h}) = \frac{1}{A}Q_{in} - \frac{n}{A}\sqrt{h}$$

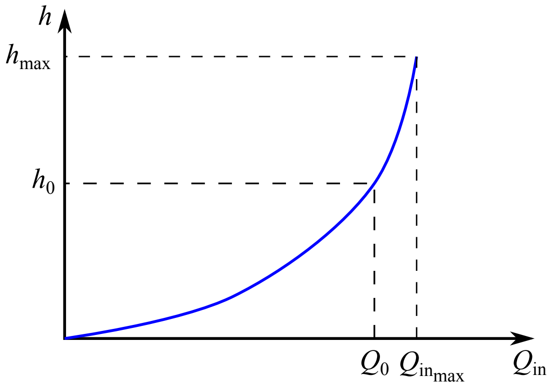
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{A}Q_{in} - \frac{n}{A}\sqrt{h} \qquad \text{- Process model} \tag{2}$$

2 Linearization

a) Steady-state $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$

$$\dot{h} + \frac{n}{A}\sqrt{h} - \frac{1}{A}Q_{in} = 0$$
$$\dot{h} = k_1 Q_{in} - k_2\sqrt{h}$$

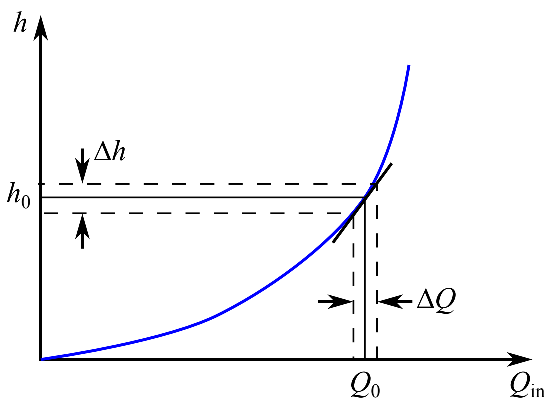
$$F(h, \dot{h}, Q_{in}) = 0$$
$$h = f(h, Q_{in})$$
$$k_1 = \frac{1}{A}$$
$$k_2 = \frac{n}{A}$$



$$\dot{h} = 0 \quad \Rightarrow \quad Q_{in} = Q_{out}$$
$$k_1 Q_{in} = k_2 Q_{out}$$
$$Q_{in} \equiv \sqrt{h}$$
$$Q_{in}^2 = h$$

$$h = f(h, Q_{in})$$
$$f(Q_{in_0}, h_0) = 0 \quad \text{steady state}$$

At a working point h_0 we want to linearize the function



$$Q_{in} = Q_{in_0} + \Delta Q_{in}$$
$$h = h_0 + \Delta h$$

$$f(Q_{in_0} + \Delta Q_{in}, h_0 + \Delta h_0) = \overbrace{f(Q_{in_0}, h_0)}^{0} + \left.\frac{\partial f}{\partial Q_{in}}\right|_{\substack{Q_{in_0} \\ h_0}} \cdot \Delta Q_{in} + \left.\frac{\partial f}{\partial h}\right|_{\substack{h_0 \\ Q_{in_0}}} \cdot \Delta h$$

$$\Delta\dot{h} = k_1 \Delta Q_{in} + k_2 \frac{1}{2} \frac{1}{\sqrt{h_0}} \cdot \Delta h = k_1 \Delta Q_{in} - \frac{k_2}{2\sqrt{h_0}} \cdot \Delta h \qquad \text{linearized equation} \tag{3}$$

3 Laplace domain representation $W(s) = \dfrac{K}{1 + T \cdot s}$:

$$\begin{aligned} \frac{\mathrm{d}\Delta h}{\mathrm{d}t} + \frac{k_2}{2\sqrt{h_0}} \cdot \Delta h &= k_1 \Delta Q_{in} \\ y + T \cdot s \frac{\mathrm{d}y}{\mathrm{d}t} &= Ku \end{aligned} \qquad \Rightarrow \qquad \Delta h + \underbrace{\frac{2\sqrt{h_0}}{k_2}}_{T \cdot s} \frac{\mathrm{d}\Delta h}{\mathrm{d}t} = \underbrace{\frac{2k_1\sqrt{h_0}}{k_2}}_{K} \Delta Q_{in}$$

$$W = \frac{\frac{2k_1\sqrt{h_0}}{k_2}}{1 + \frac{2\sqrt{h_0}}{k_2} s} = \frac{\frac{2\frac{1}{A}\sqrt{h_0}}{\frac{n}{A}}}{1 + \frac{2\sqrt{h_0}}{\frac{n}{A}} s} = \frac{\frac{2\sqrt{h_0}}{n}}{1 + \frac{2A\sqrt{h_0}}{n} s} \tag{4}$$