Dynamic systems on time scales

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The Seventh Annual Conference of the National Doctoral School in Information and Communication Technologies
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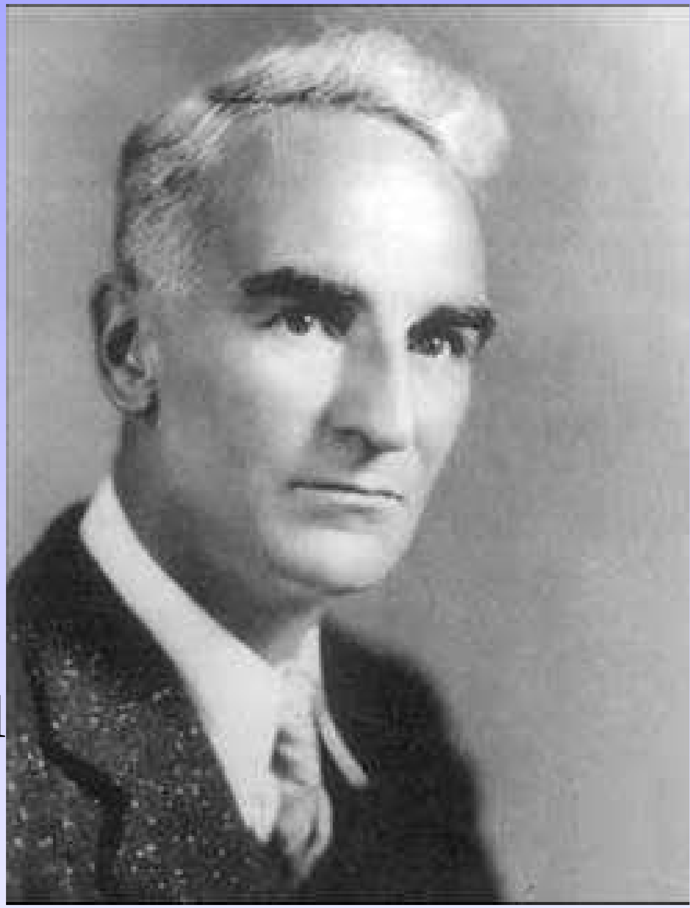
Contents

Background

A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

"*Men of Mathematics*", E.T. Bell



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Examples of time scales:

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discrete time

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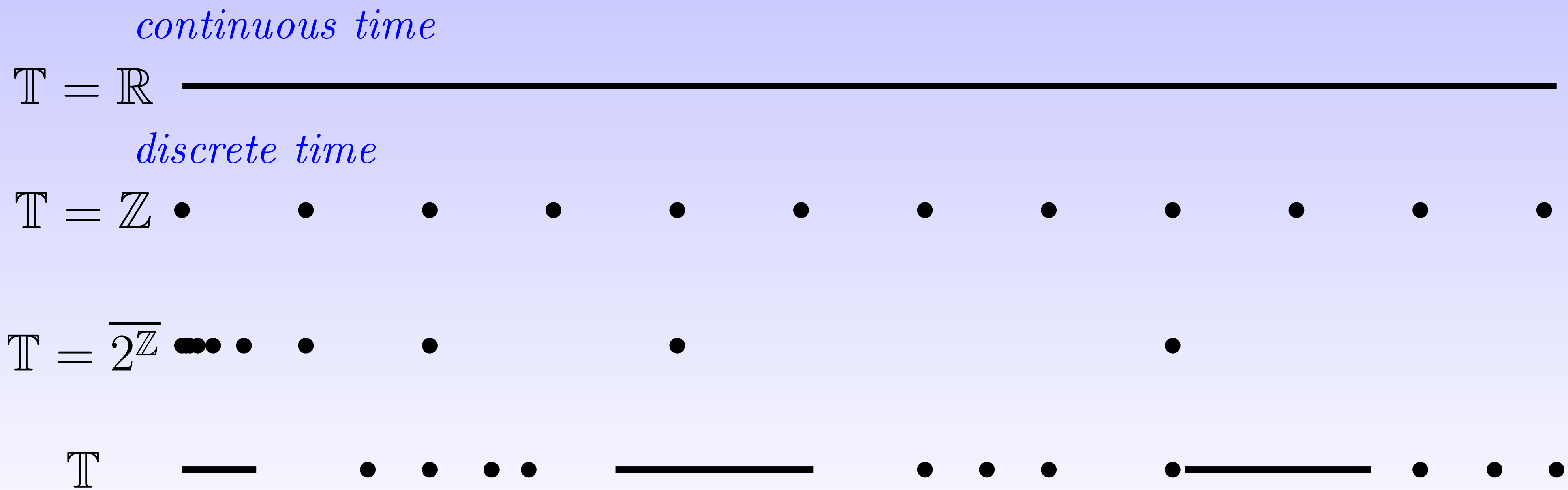

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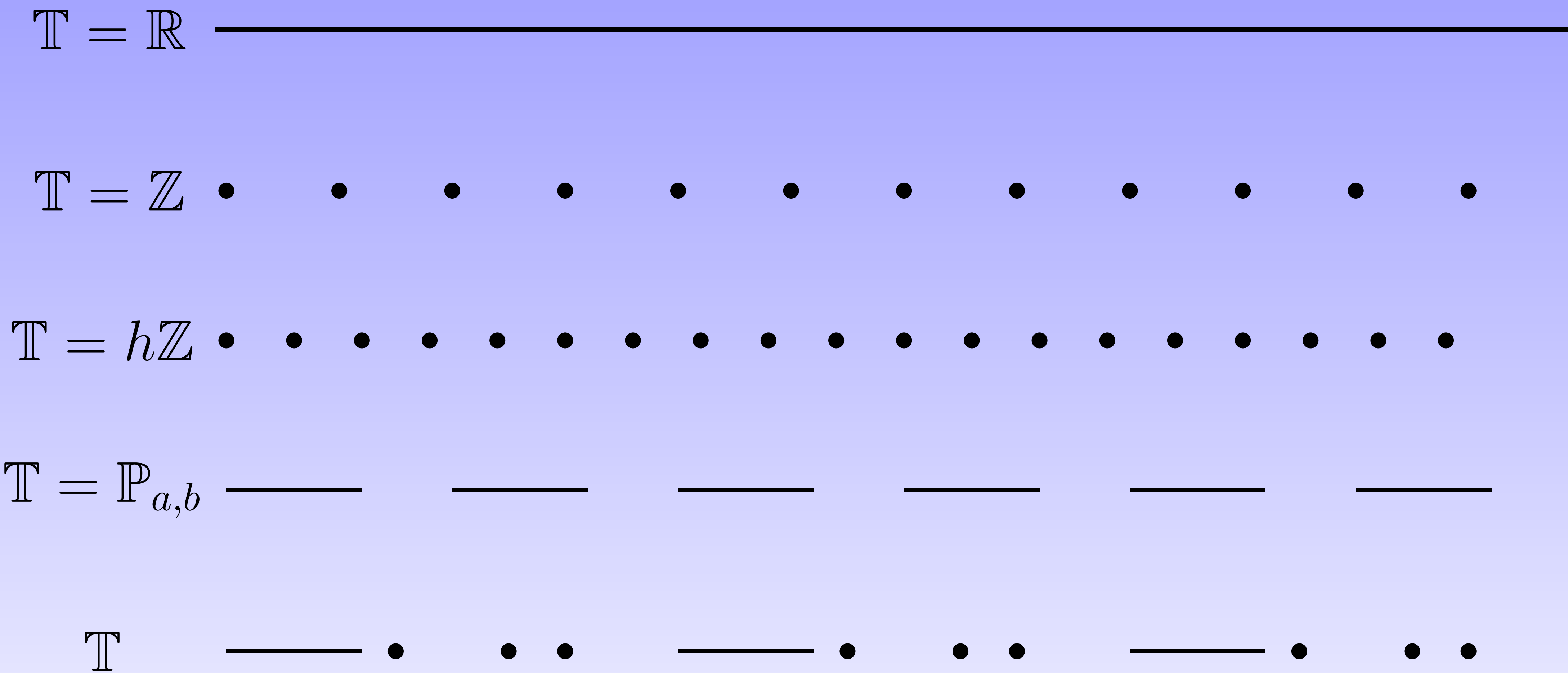
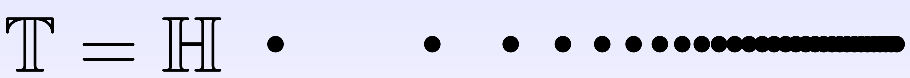
- ☞ Time Scales Research Group from Baylor University

  http://web.ecs.baylor.edu/faculty/marks/Research/TimeScales/index.htm

Time scales & applications





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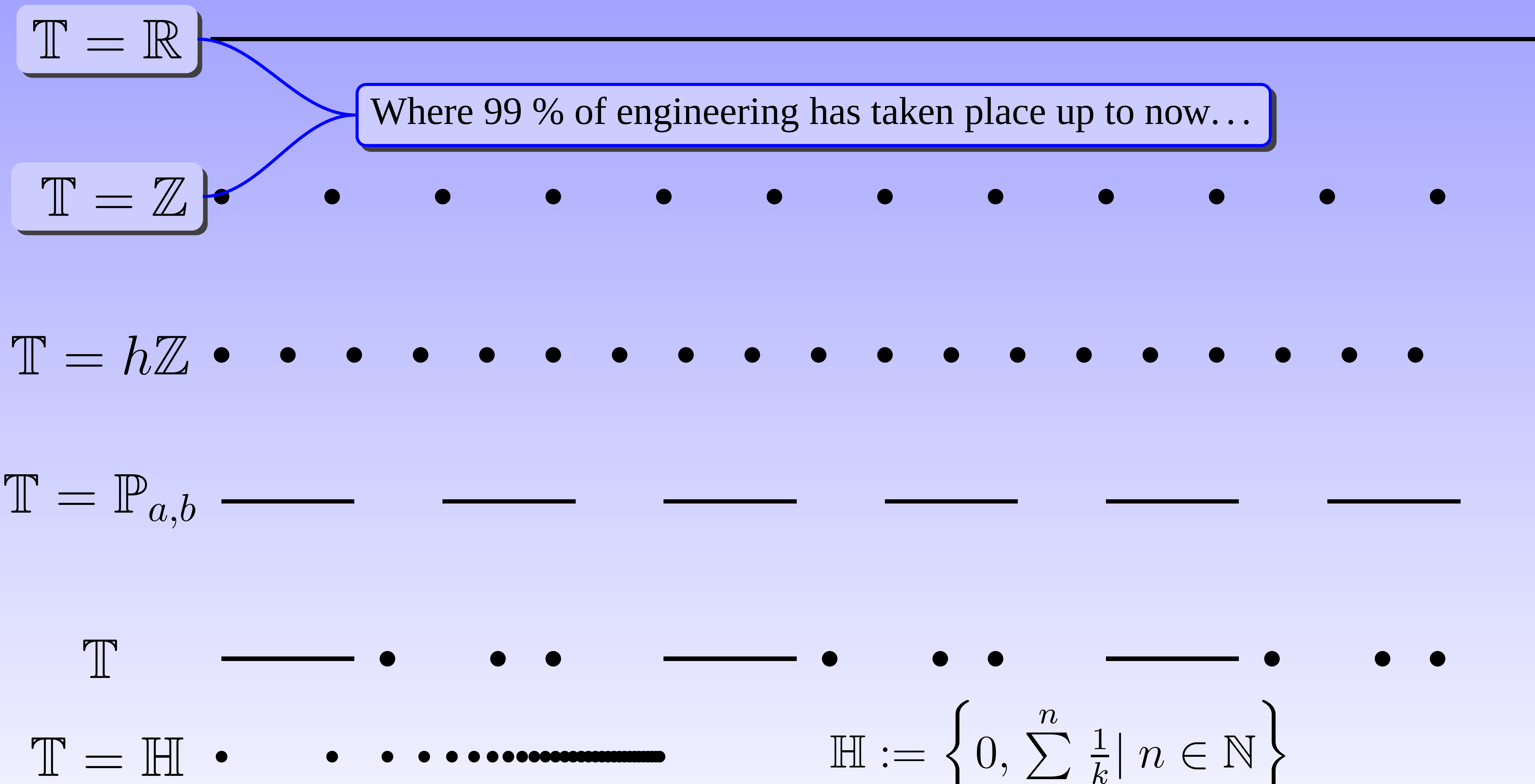
Time scales & applications



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Time scales & applications

$\mathbb{T} = \mathbb{R}$

Where 99 % of engineering has taken place up to now. . .

$\mathbb{T} = \mathbb{Z}$

$\mathbb{T} = h\mathbb{Z}$

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The real world that engineers have tried to avoid.

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Operators related to time scales

☞ *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$$

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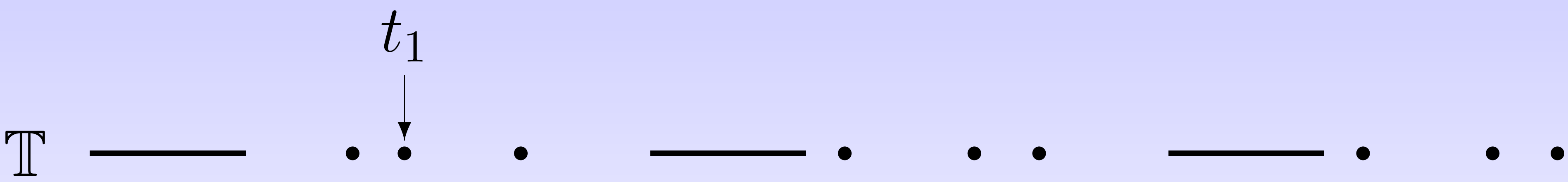

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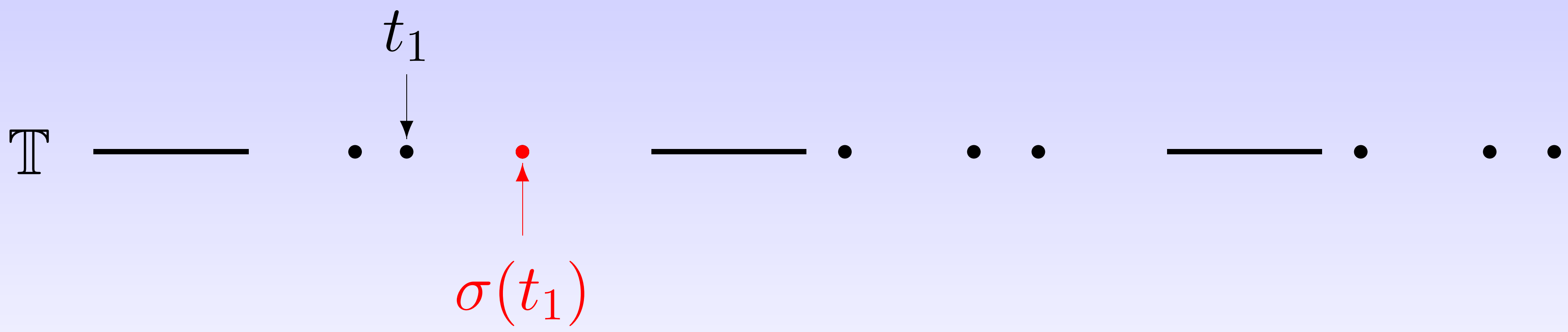

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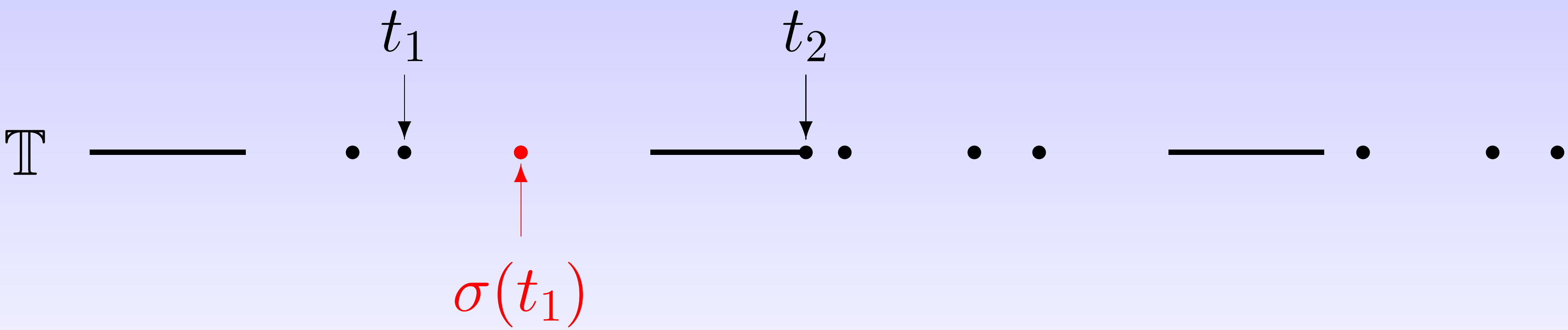

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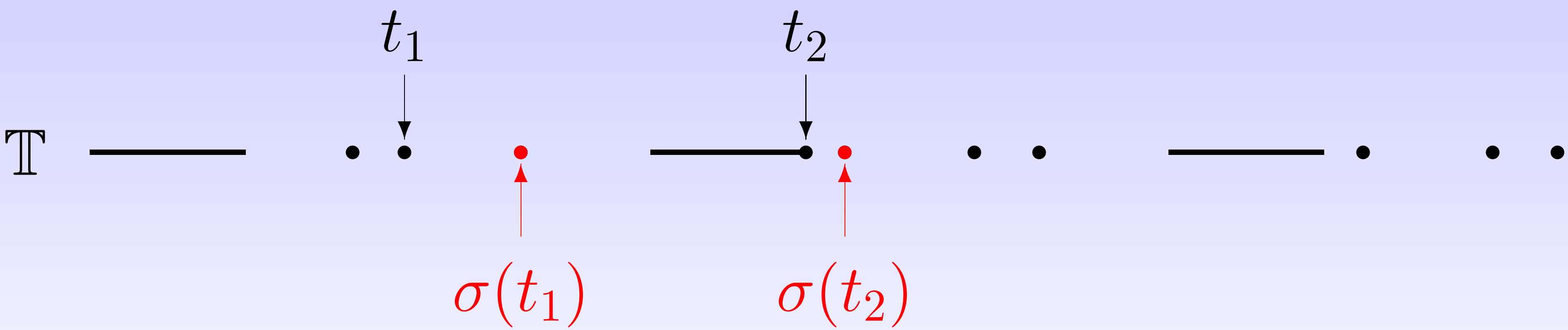

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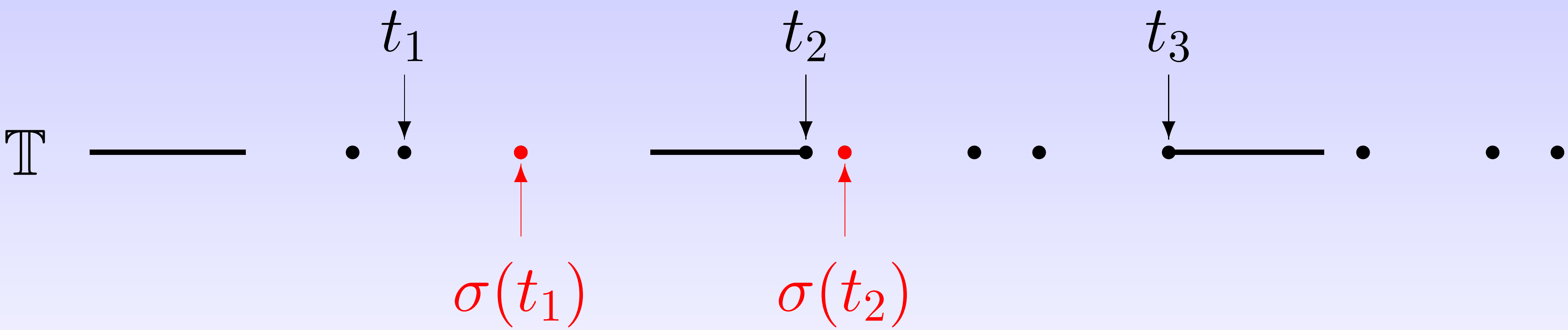

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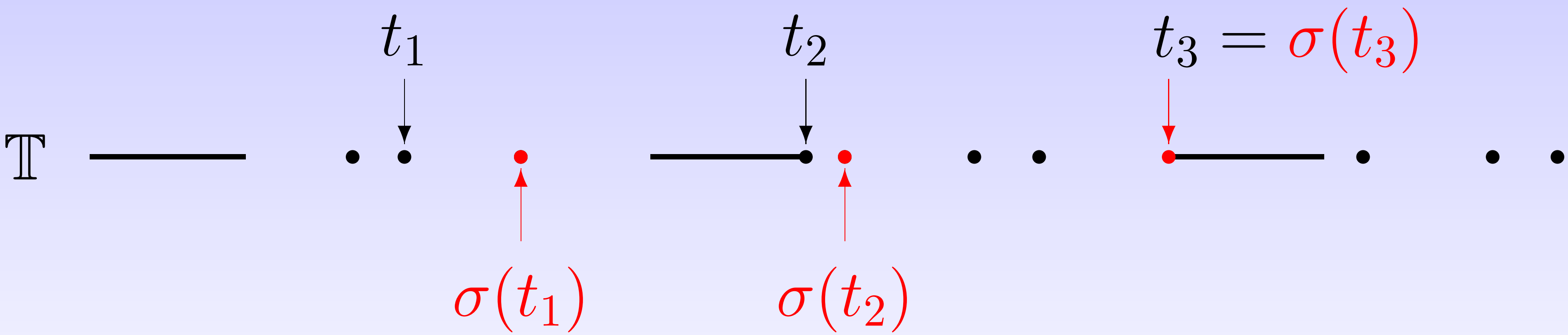

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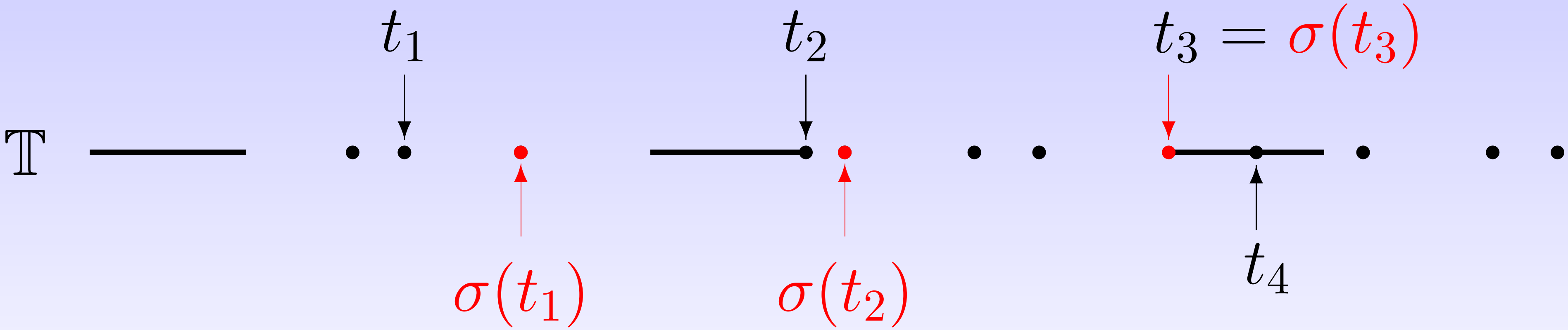

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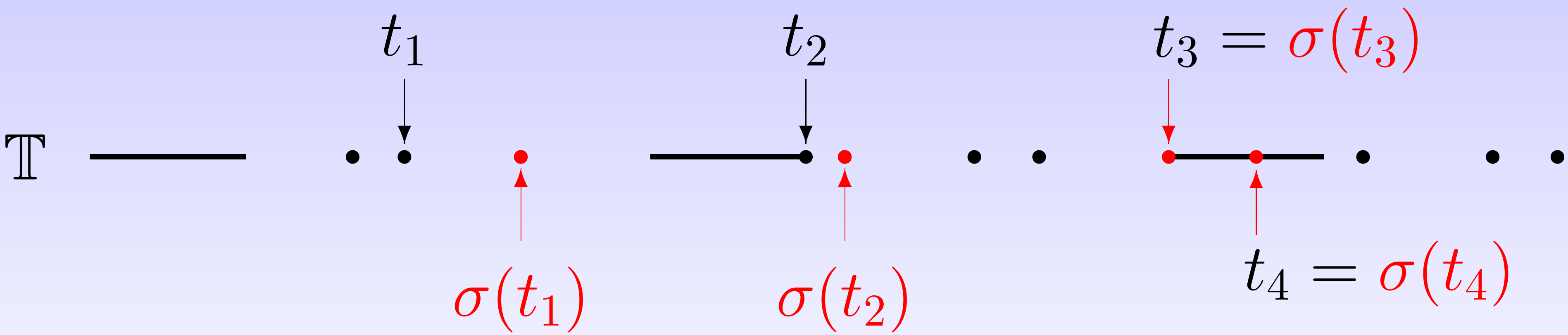

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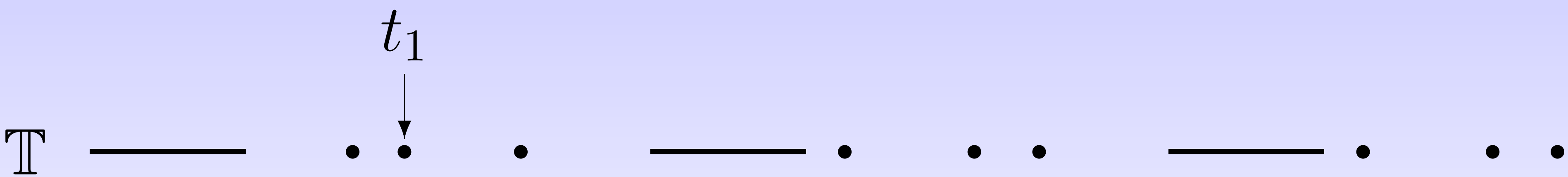

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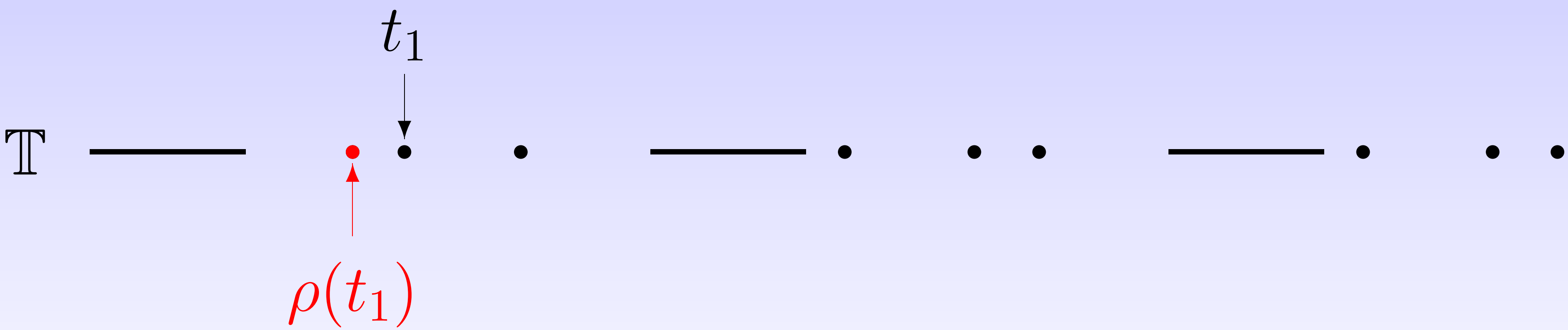

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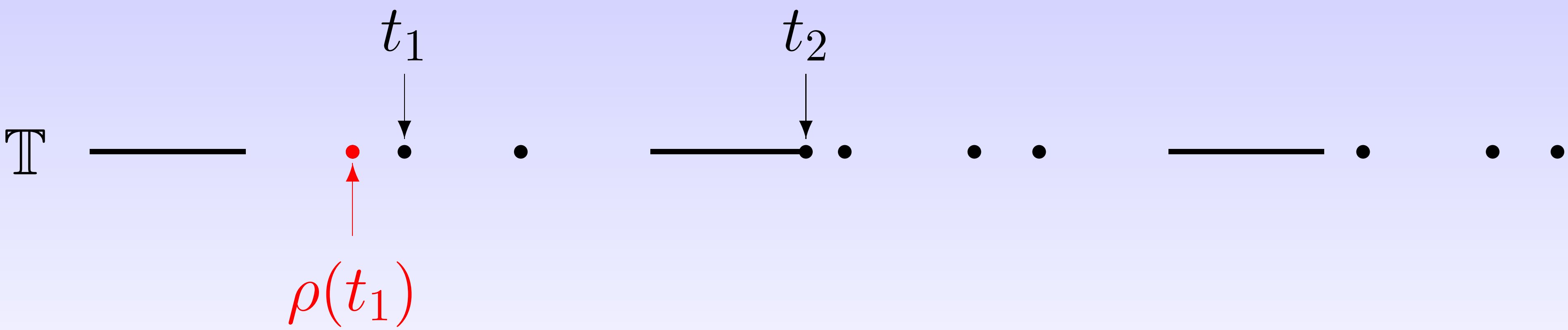

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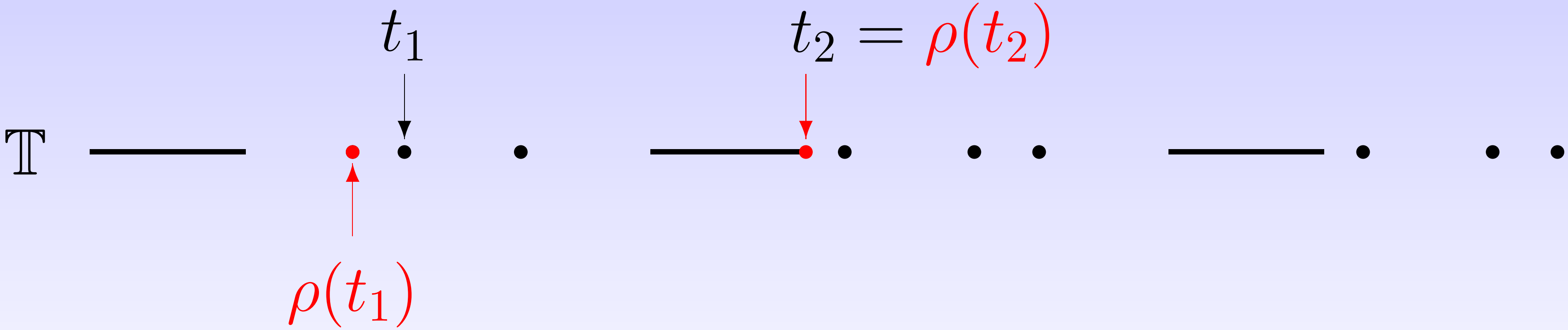

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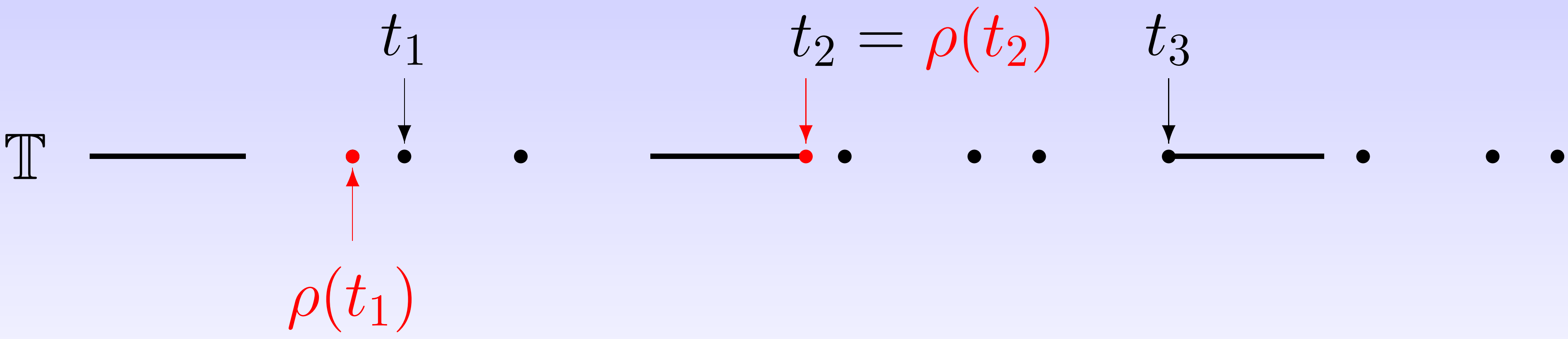

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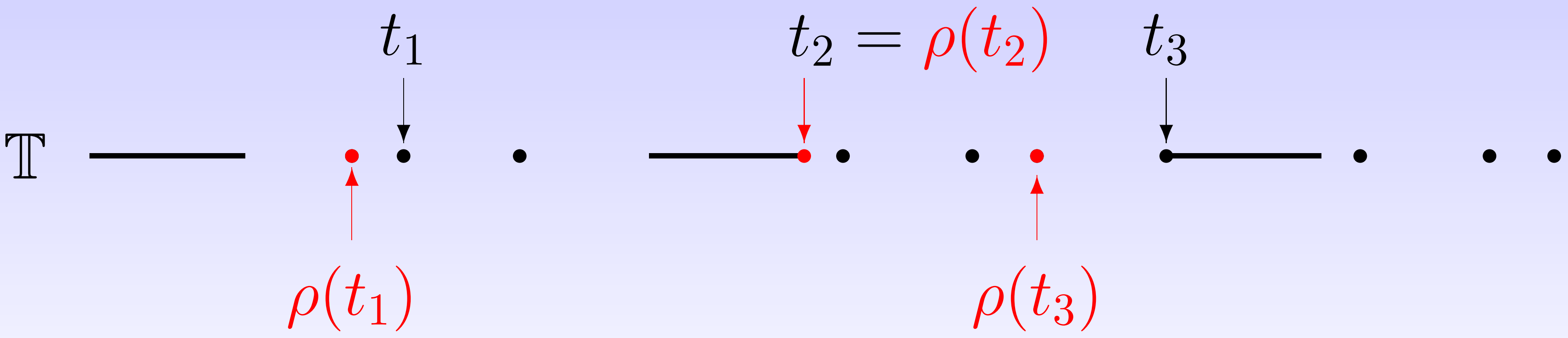

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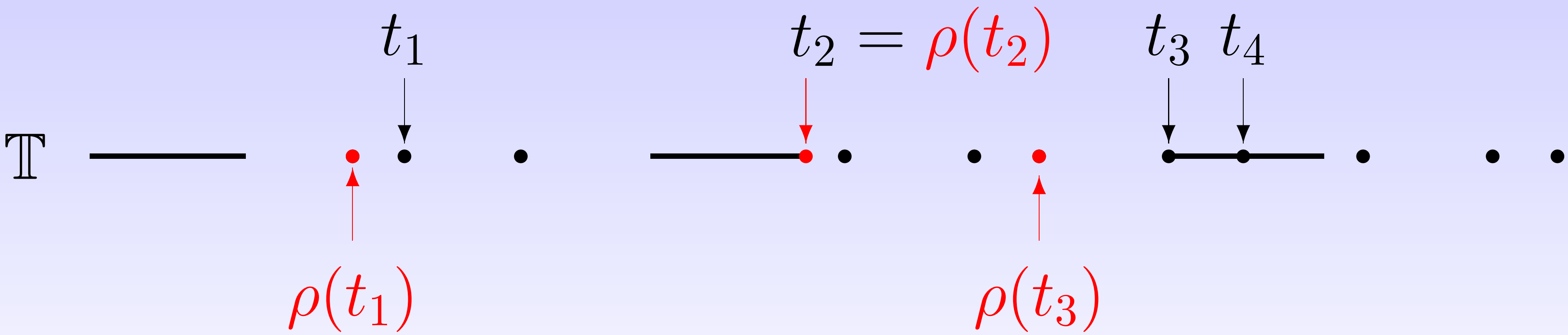

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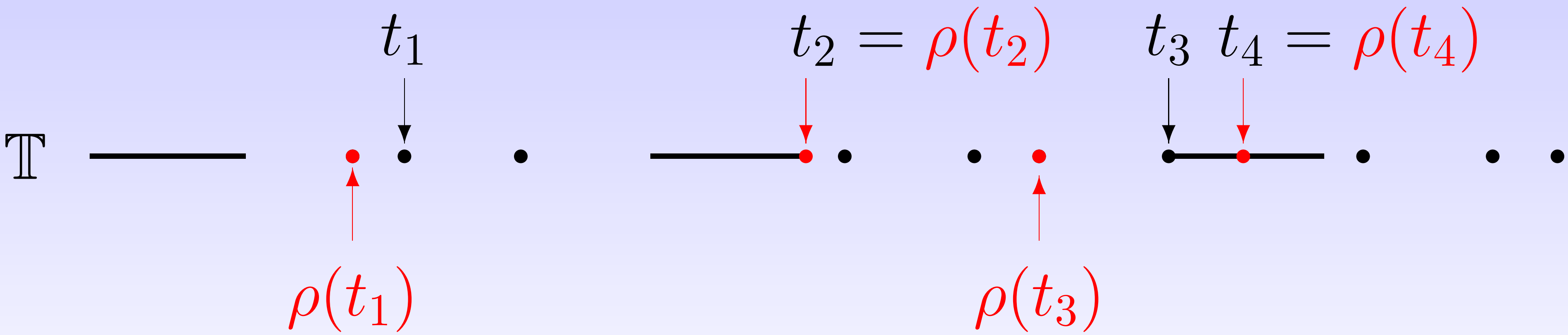

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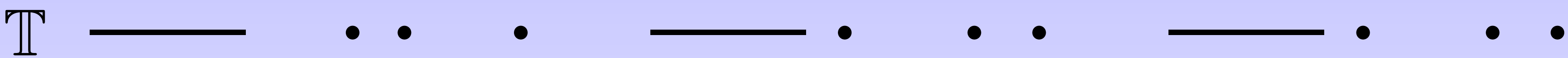

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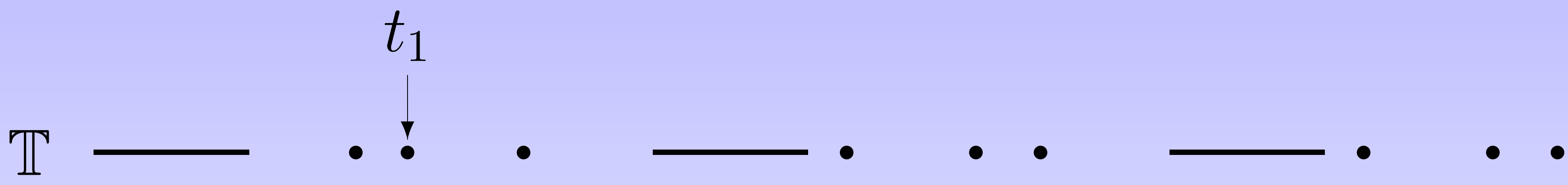

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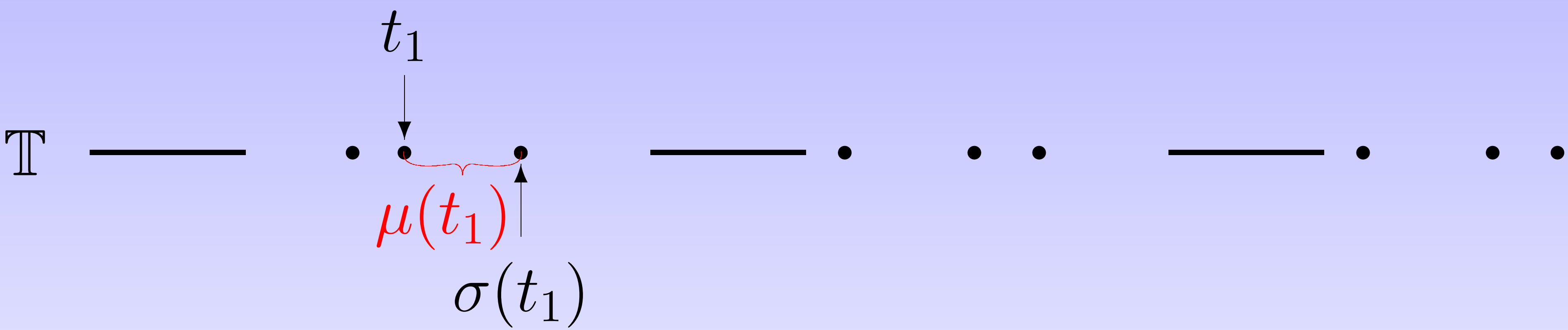

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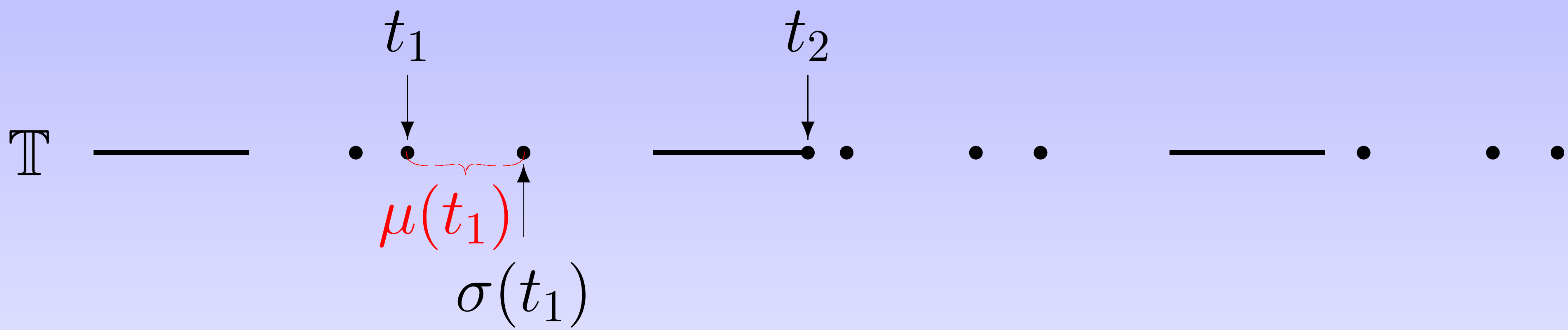

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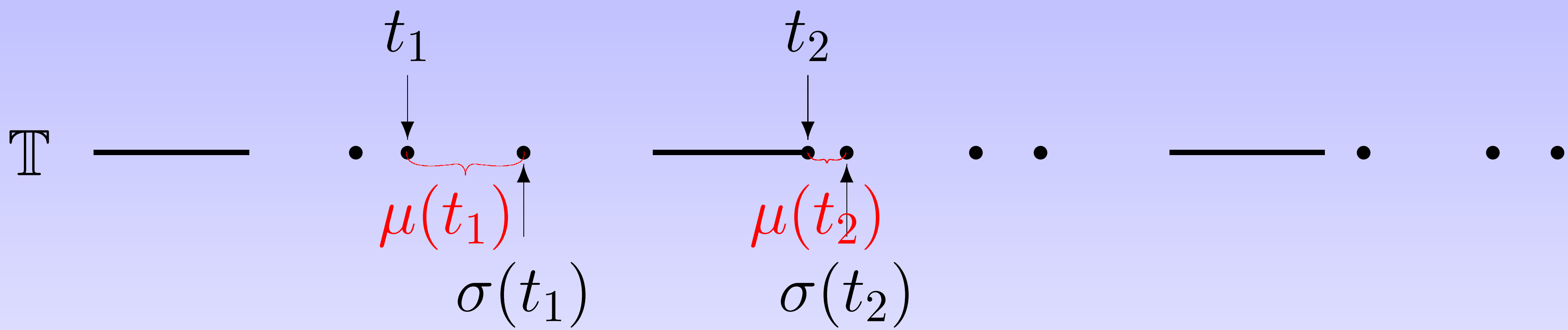

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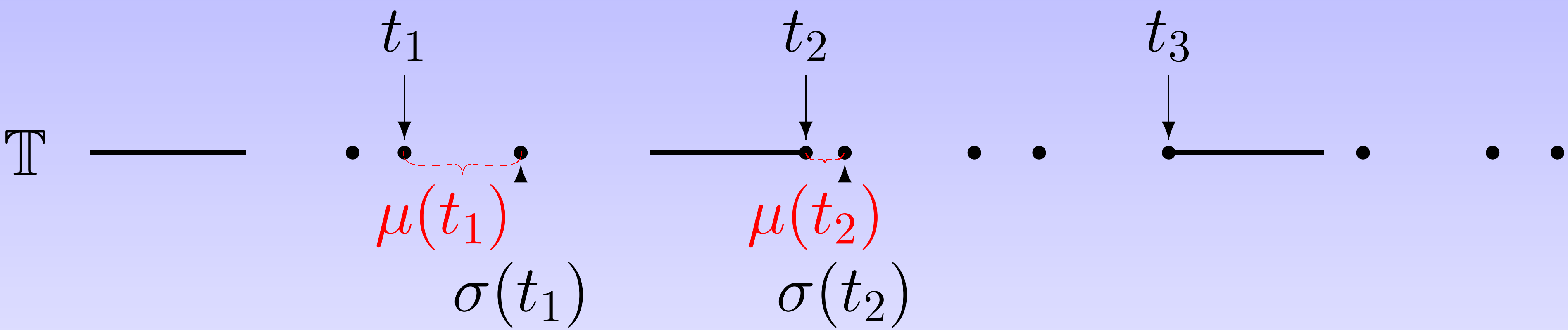

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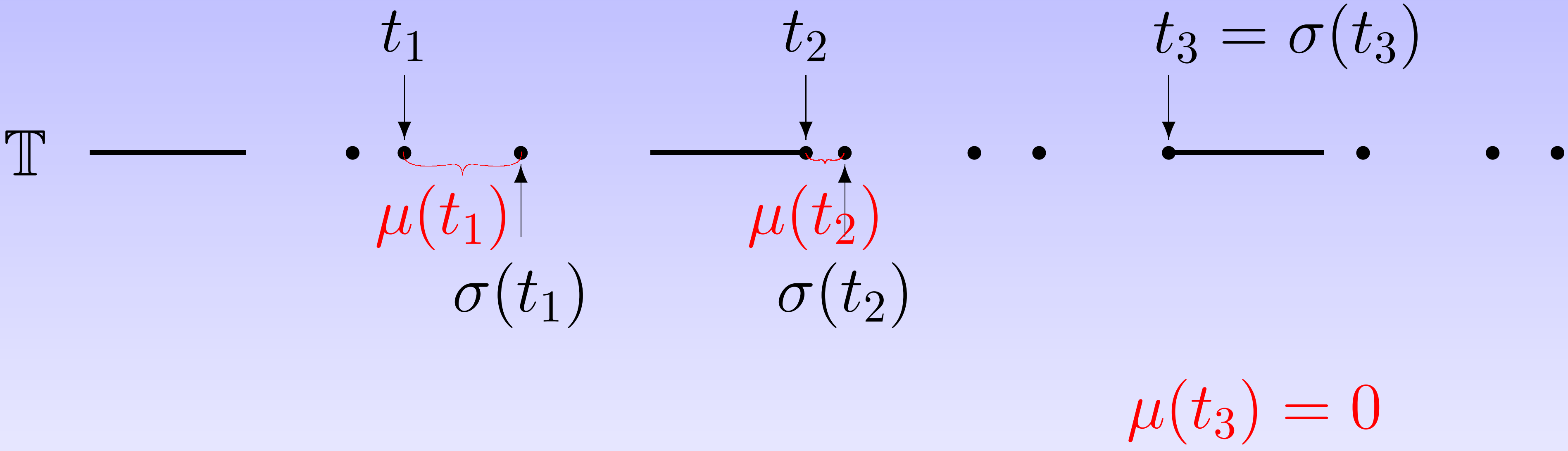

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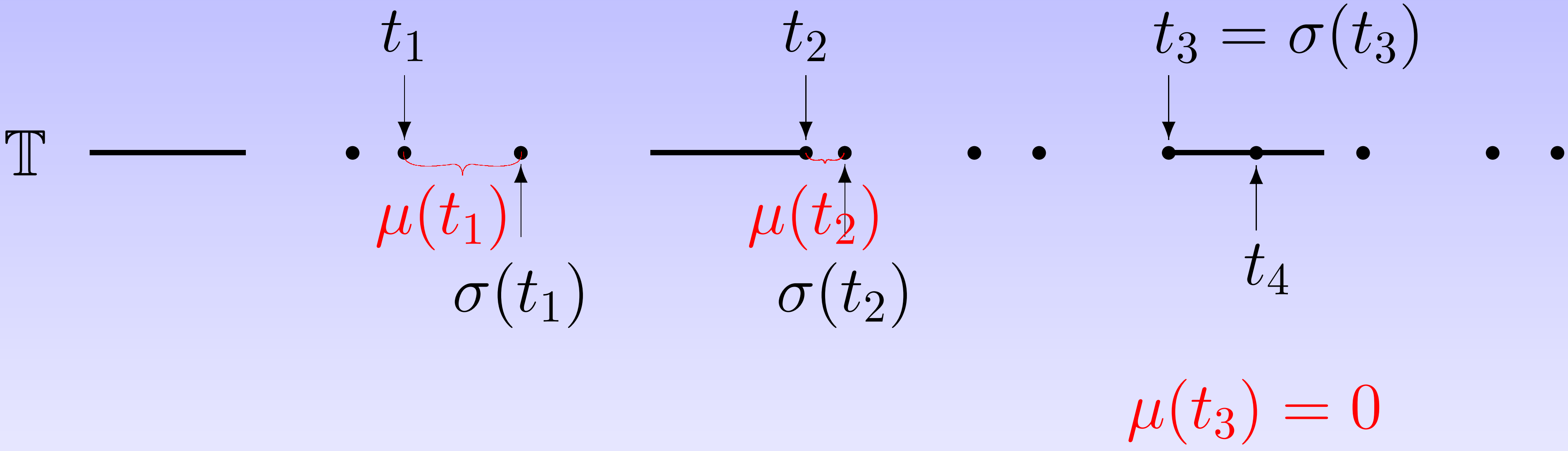

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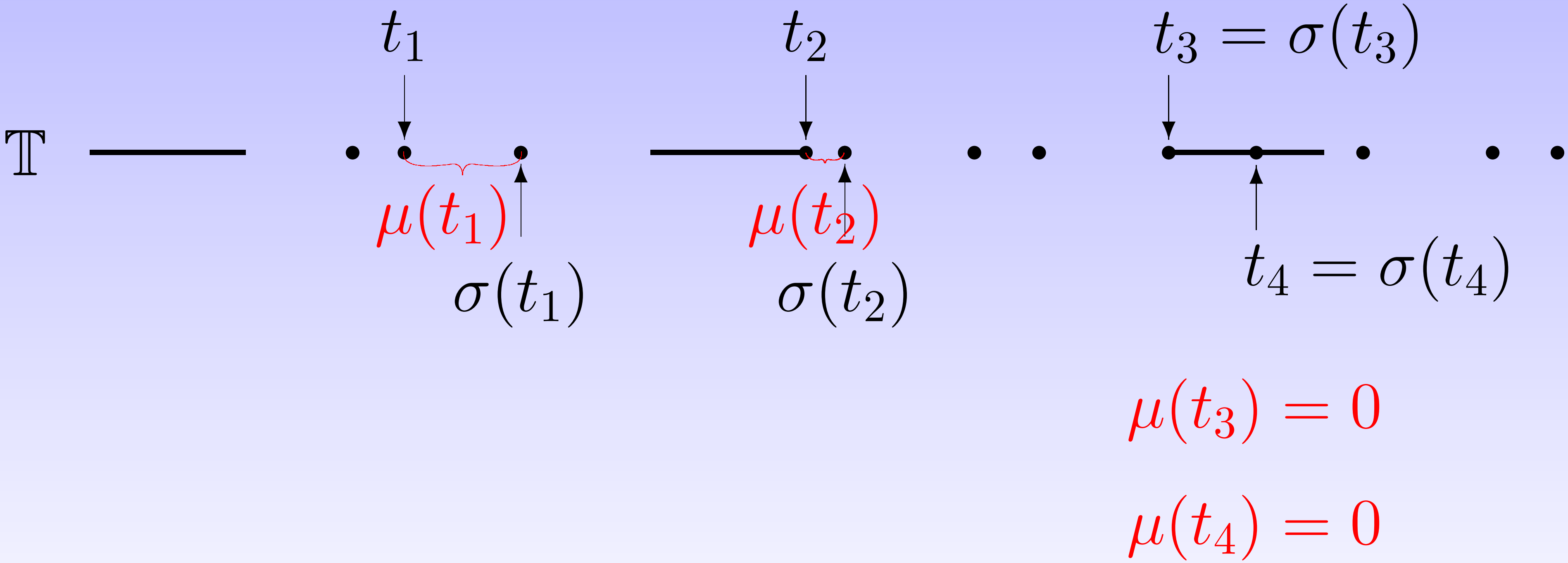

Example

$\mathbb{T} = \mathbb{R}$

Then for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} \mid s > t\} = \inf\ (t, +\infty)\ = t,$$

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Hence

$$\rho(t) = t = \sigma(t)$$

and

$$\mu(t) = 0.$$

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Example

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Then for every $t \in h\mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} \mid s > t\} = \inf \ \{t + h, t + 2h, \ldots\} \ = t + h,$$

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Therefore

$$\rho(t) = t - h, \qquad \sigma(t) = t + h$$

and

$$\mu(t) = h.$$

Examples

☞ Let $1 \neq q > 0$. If $\mathbb{T} = \{q^n \mid n \in \mathbb{Z}\} \cup \{0\}$, then

$$\sigma(t) = qt, \quad \rho(t) = \frac{t}{q} \quad \text{and} \quad \mu(t) = (q-1)t.$$

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- ☞ If $\mathbb{T} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, then

$$\sigma(t) = \frac{t}{1-t}, \quad \rho(t) = \frac{t}{1+t} \quad \text{and} \quad \mu(t) = \frac{t^2}{1-t}.$$

Examples

☞ Let $1 \neq q > 0$. If $\mathbb{T} = \{q^n \mid n \in \mathbb{Z}\} \cup \{0\}$, then

$$\sigma(t) = qt, \quad \rho(t) = \frac{t}{q} \quad \text{and} \quad \mu(t) = (q-1)t.$$

☞ If $\mathbb{T} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, then

$$\sigma(t) = \frac{t}{1-t}, \quad \rho(t) = \frac{t}{1+t} \quad \text{and} \quad \mu(t) = \frac{t^2}{1-t}.$$

☞ If $\mathbb{T} = \left\{\frac{n}{2} \mid n \in \mathbb{N}_0\right\}$, then

$$\sigma(t) = t + \frac{1}{2}, \qquad \rho(t) = t - \frac{1}{2} \quad \text{and} \quad \mu(t) = \frac{1}{2}.$$

Examples

☞ If $\mathbb{T} = \left\{n^2 \mid n \in \mathbb{N}_0\right\}$, then

$$\sigma(t) = \left(\sqrt{t}+1\right)^2, \ \rho(t) = \left(\sqrt{t}-1\right)^2, \ \mu(t) = 1 + 2\sqrt{t}.$$

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☞ If $\mathbb{T} = \{ \sqrt[3]{n} \mid n \in \mathbb{N}_0 \}$, then

$$\sigma(t) = \sqrt[3]{t^3+1}, \rho(t) = \sqrt[3]{t^3-1}, \mu(t) = \sqrt[3]{t^3+1} - t.$$

Example

$\mathbb{T} = \mathbb{P}_{a,b}$

If $\quad t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a),\quad$ then

$$\sigma(t) = t.$$

If $\quad t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\},\quad$ then

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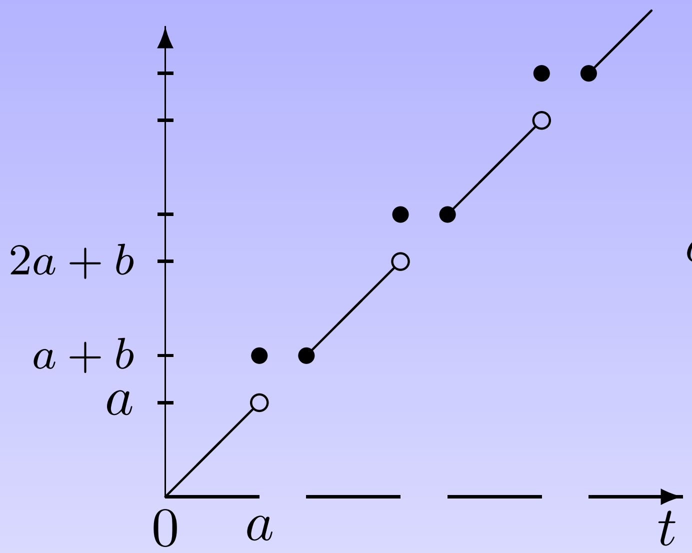

$$\sigma(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ t+b, & t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\} \end{cases}$$

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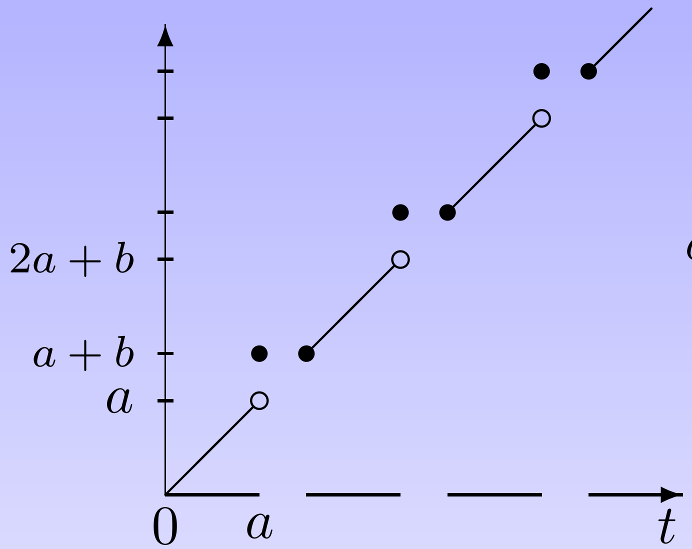

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$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ b, & \text{if } t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}. \end{cases}$$

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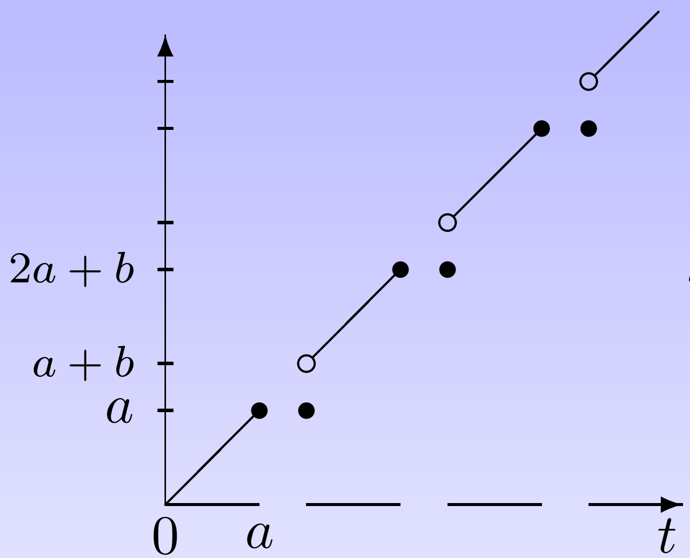

$$\rho(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} (k(a+b), k(a+b)+a] \\ t-b, & t \in \bigcup_{k=0}^{\infty} \{k(a+b)\} \end{cases}$$

Example

$\mathbb{T} = \mathbb{P}_{1,1}$

Then

$$\rho(1) = 1 \qquad \sigma(1) = 2$$

so

$$(\sigma \circ \rho)(1) = \sigma(\rho(1)) = 2 \neq 1.$$

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Homogenous time scales

A time scale $\mathbb{T}$ is called *homogeneous* if

$$\mu \equiv \text{const.}$$

Time scales

$$\mathbb{T} = \mathbb{R} \qquad \text{and} \qquad \mathbb{T} = h\mathbb{Z}, \quad h > 0$$

are both homogenous.

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Then

$$\boxed{\mu \equiv 0} \qquad \text{and} \qquad \boxed{\mu \equiv h}$$

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Regular time scales

A time scale $\mathbb{T}$ is called ***regular*** if the following two conditions are satisfied simultaneously for all $t \in \mathbb{T}$:

$$(i)\ \ \sigma(\rho(t)) = t, \qquad (ii)\ \rho(\sigma(t)) = t.$$

Every homogeneous time scale is regular, since

$$\mu \equiv \text{const} = h,\ \sigma(t) = t + h \text{ and } \rho(t) = t - h.$$

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Time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, $h > 0$ are both homogenous and regular.

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Time scales

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} \ \text{ and } \ \mathbb{T} = (-\infty, 0] \cup \left\{\tfrac{1}{k} \middle|\ k \in \mathbb{N}\right\} \cup \{2k |\ k \in \mathbb{N}\}$$

are both regular, but not homogenous.

Delta derivative

Delta derivative of $f : \mathbb{T} \to \mathbb{R}$, denoted by f^{Δ}, can be defined as the extension of standard time-derivative in the continuous-time case.

- If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

- If $\mathbb{T} = \mathbb{Z}$, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{1} = f(t+1) - f(t) =: \triangle f(t).$$

Delta derivative

☞ If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $f : \mathbb{T} \to \mathbb{R}$ is always delta differentiable on $t \in \mathbb{T} \setminus \{0\}$ and for $t \in \mathbb{T} = q^{\mathbb{Z}}$

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t}.$$

Moreover

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s},$$

if only this limit exists.

Examples of delta derivatives

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$.

☞ $f(t) = t$ → $f^{\Delta}(t) \equiv 1$

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- ☞ $f(t) = t^n$ → $f^{\Delta}(t) = \sum_{k=1}^{n} t^{n-k} \left(\sigma(t)\right)^{k-1}$

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- $f(t) = t^n$ → $f^{\Delta}(t) = \sum_{k=1}^{n} t^{n-k} (\sigma(t))^{k-1}$
- $f(t) = \frac{1}{t}$ → $f^{\Delta}(t) = -\frac{1}{t\sigma(t)}$ $\quad t \neq 0.$
- $f(t) = \sqrt{t}$ → $f^{\Delta}(t) = \frac{1}{\sqrt{\sigma(t)}+\sqrt{t}}$ $\quad t \neq 0.$

Properties of delta derivatives

- ☞ $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$
- ☞ $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$
- ☞ $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$
- ☞ $$\begin{aligned}(f\cdot g)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)\\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$
- ☞ $\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$

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Some applications of time scales

☞ $N(t)$ - the amount of plants at time t.

During the months of April until September:

$$N' = N.$$

October 1^{st}:	all plants suddenly die
	but the seeds remain in the ground
April 1^{st}:	seeds start growing again
	with N now being doubled.

We can model this situation using the time scales

$$\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1],$$

where

$t = 2k$ is April 1st of k^{th} year,

$t = 2k+1$ is October 1st of k^{th} year.

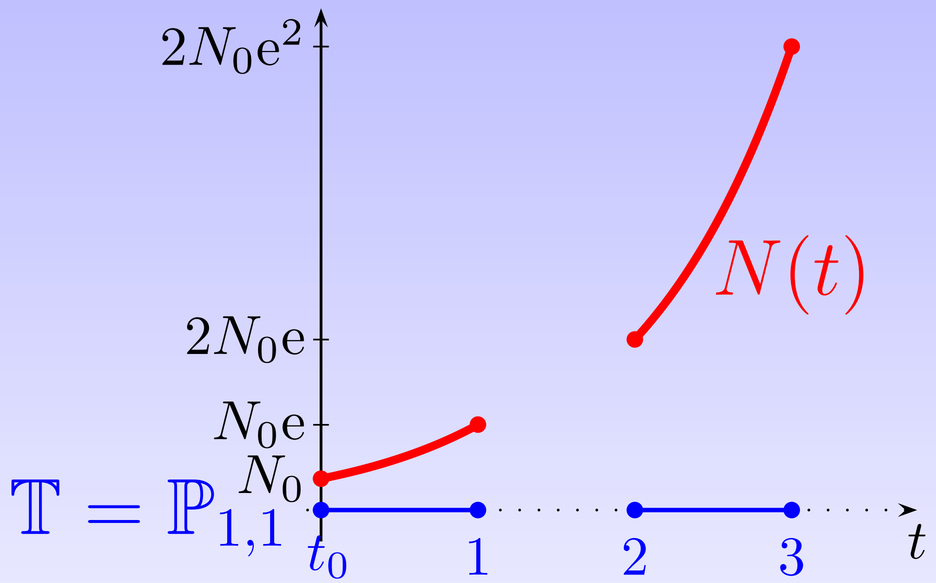
Then

$$N^{\Delta}(t) = N(t), \qquad t \in \mathbb{P}_{1,1}$$

where $N(t)$ is the amount of plants at time t.

Then

$$N(t) = 2^k e^{t-k} N_0, \quad \text{for } 2k \leqslant t \leqslant 2k+1, \;\; k \in \mathbb{N}_0.$$

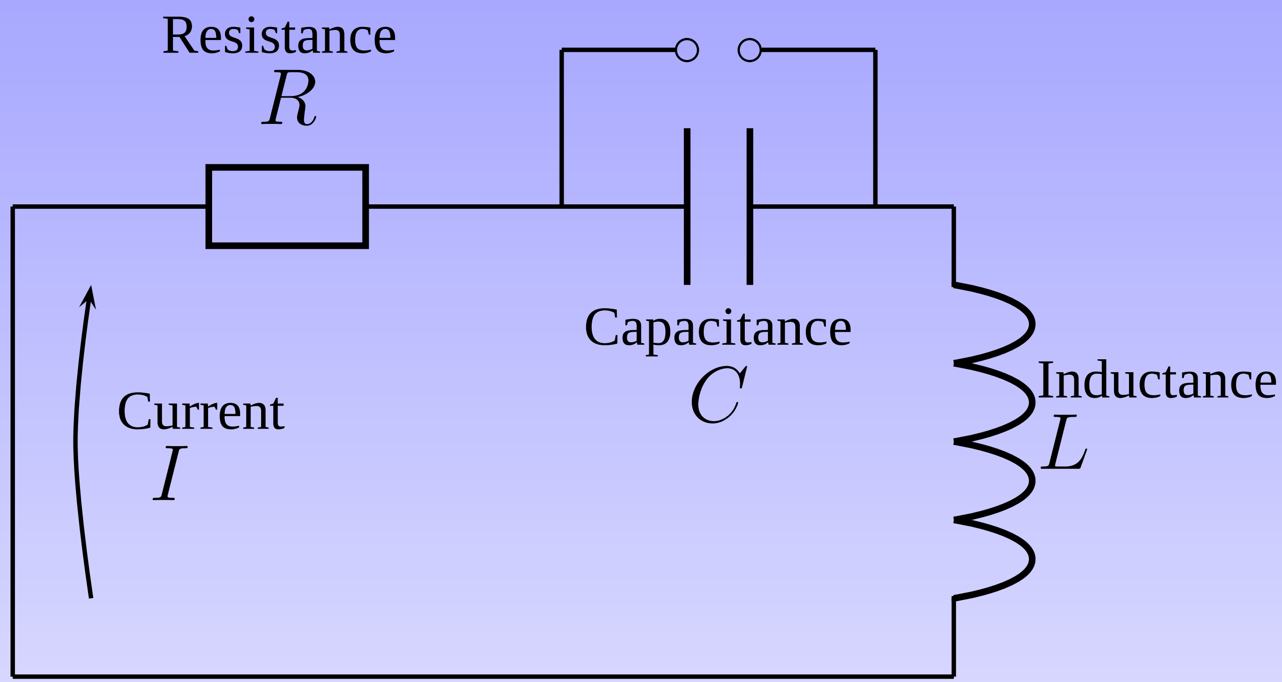


The following examples can be modeled with similar time scales:

- ☞ insect population model, which are discrete in season, die out in say winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population
- ☞ some specific example is the cicada *Magicicada septendecim* which lives as a larva for 17 years as an adult for perhaps a week
- ☞ another example is the common mayfly *Stenonema canadense* which lives as a larva for a years as an adult for less than a day.

Electric circuit

☞



Suppose we decharge the capacitor periodically every time unit and assume that the decharging takes $\delta > 0$ (but small) time units.

Then this simulation can be modeled using time scale

$$\mathbb{T} = \mathbb{P}_{1-\delta,\delta} = \bigcup_{k=0}^{\infty} [k, k+1-\delta].$$

If $Q(t)$ is the total charge on the capacitor at time t and $I(t)$ is the current as a function of time t, then we have

$$Q^{\Delta}(t) = \begin{cases} bQ(t), & \text{if } t \in \bigcup_{k\in\mathbb{N}} \{k-\delta\} \\ I, & \text{otherwise.} \end{cases}$$

$$I^{\Delta}(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k\in\mathbb{N}} \{k-\delta\} \\ -\frac{1}{LC}Q - \frac{R}{L}I, & \text{otherwise,} \end{cases}$$

where b is a constant satisfying $-1 < b\delta < 0$.

Economical application of time scales

By varying the time and interest rate of an account we are able to find out how much money will be in the account after time $t \in \mathbb{T}$.

☞ Suppose we open an account, initially at time $t = 1$, invest \$2,000 and add an additional \$2,000 every time interest is compounded.
Let $\mathbb{T} = q^{\mathbb{N}_0}$ where $t_0 = q_0 = 1$. The interest is compounded every $t = q^n$ years and at a rate of t%.
How much will we have in the account at time t?
How much would we have at time $t = (1,5)^5$?

Since $y(1) = 0$, let $y(t)$ be the amount of money after time $t = q^n$. Therefore

$$\begin{aligned} y(\sigma(t)) &= y(t) + 2000 + 0.01t(y(t) + 2000) \\ y(\sigma(t)) &= (1 + 0.01t)y(t) + (1 + 0.01t)2000 \end{aligned}$$

and

$$y^{\Delta}(t) = \frac{0.01}{q-1}y(t) + \frac{1+0.01t}{(q-1)t}2000.$$

So the solution of IVP for the problem is

$$y(t) = 2000 \sum_{i \in [1,t)} \left((1 + 0.01i) \prod_{s=\sigma(i)}^{\rho(t)} (1 + 0.01s) \right).$$

If $q = 1.5$ and $n = 5$ we calculate how much money is in our account after $t = (1.5)^5 \approx 7.59$, so approximately 6.59 years after our initial investment, we get

$$y((1.5)^5) \approx \$11025.71.$$

The time scales calculus unifying the continuous- and discrete-time control systems

Continuous-time control systems	*Discrete-time control systems*
$t \in \mathbb{R}$	$t \in \mathbb{Z}$
$\dot{x}(t) = f_1(x(t), u(t))$	$x(t+1) = f_2(x(t), u(t))$

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If $\mathbb{T} = \mathbb{R}$, then	If $\mathbb{T} = \mathbb{Z}$, then
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If $\mathbb{T} = \mathbb{R}$, then $\dot{x}(t) = x^{\Delta}(t).$	If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta}(t) = x(t+1) - x(t).$
Therefore $x^{\Delta}(t) = f_1(x(t), u(t))$	Therefore $x^{\Delta}(t) = f_2(x(t), u(t)) - x(t)$

The time scales calculus unifying the continuous- and discrete-time control systems

Continuous-time control systems

$$t \in \mathbb{R}$$

$$\dot{x}(t) = f_1(x(t), u(t))$$

If $\mathbb{T} = \mathbb{R}$, then

$$\dot{x}(t) = x^{\Delta}(t).$$

Therefore

$$x^{\Delta}(t) = f_1(x(t), u(t))$$

Discrete-time control systems

$$t \in \mathbb{Z}$$

$$x(t+1) = f_2(x(t), u(t))$$

If $\mathbb{T} = \mathbb{Z}$, then

$$x^{\Delta}(t) = x(t+1) - x(t).$$

Therefore

$$x^{\Delta}(t) = f_2(x(t), u(t)) - x(t)$$

$$x^{\Delta}(t) = f(x(t), u(t)) \qquad t \in \mathbb{T}$$

Solution of control systems

$x^{\Delta}(t) = u(t)$ $t \in \mathbb{T}$ and $x(0) = 1$

Solution of control systems

$$x^{\Delta}(t) = u(t) \quad t \in \mathbb{T} \quad \text{and} \quad x(0) = 1$$

Continuous-time control systems *Discrete-time control systems*

$t \in \mathbb{R}$ $t \in \mathbb{Z}$

$$\dot{x}(t) = u(t)$$

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Continuous-time control systems | *Discrete-time control systems*

$t \in \mathbb{R}$: $\dot{x}(t) = u(t)$

$t \in \mathbb{Z}$: $x(t+1) = u(t) + x(t)$

Then

$$x(t) = \int_0^t u(\tau)d\tau + 1, \quad t \geqslant 0$$

where input u is the integrable function applied for the system.

$$x(t) = \sum_{n=0}^{t-1} u(n) + 1, \quad t \in \mathbb{N}$$

where $u(n)$, $n \in \mathbb{N}_0$ is a sequence of controls applied for the system.

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Integration

A function $F : \mathbb{T} \to \mathbb{R}$ is called a ***antiderivative of*** $f : \mathbb{T} \to \mathbb{R}$ if it satisfies $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}$.

$$\int f(t)\Delta t = F(t) + \text{Const}$$

- ***indefinite integral of*** f

For $s, t \in \mathbb{T}$

$$\int_r^s f(t)\Delta t = F(s) - F(r)$$

- ***Cauchy integral of*** f

Cauchy integral of f

☞ Let $\mathbb{T} = \mathbb{R}$. Then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t \quad - \quad \text{Riemann integral.}$$

☞ Let $\mathbb{T} = \mathbb{Z}$. Then, for $a < b$, $a, b \in \mathbb{Z}$,

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k) \quad - \quad \text{discrete summation.}$$

☞ Let $\mathbb{T} = \overline{2^{\mathbb{Z}}}$. Then

$$\int_a^b f\Delta t = \sum_{t\in[a,b)\cap\mathbb{T}} t f(t).$$

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Examples

If $\mathbb{T} = \mathbb{R}$, then for $t \in \mathbb{R}$

$$x^\Delta(t) = t,\ x(0) = x_0$$

$$\Updownarrow$$

$$x(t) = \frac{1}{2}t^2 + x_0.$$

If $\mathbb{T} = \mathbb{Z}$, then for $t \in \mathbb{Z}$

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$$x(t) = \frac{t(t-1)}{2} + x_0.$$

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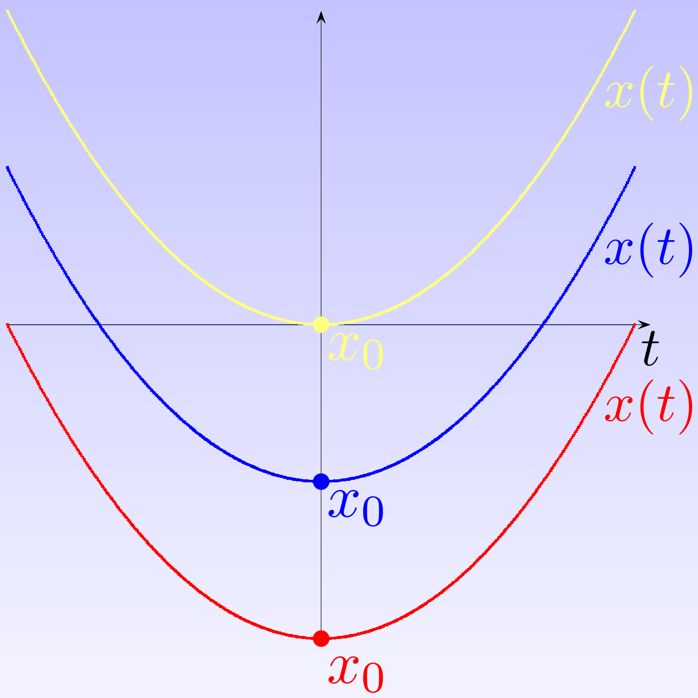

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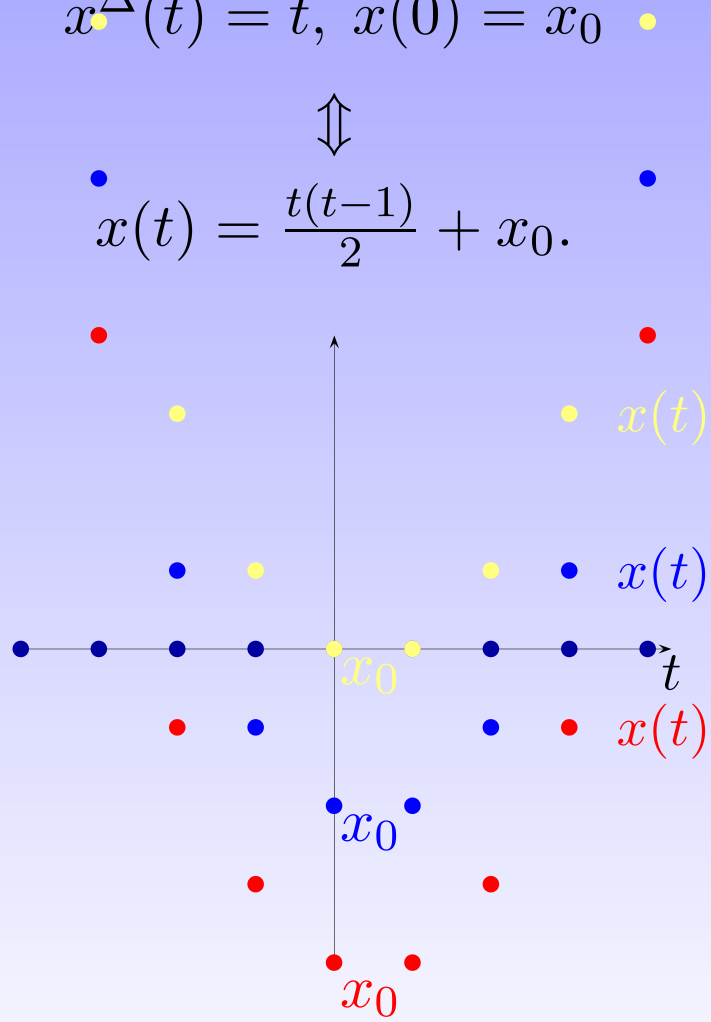

Example

If $\mathbb{T} = \mathbb{P}_{1,1}$, then for $t \in \mathbb{P}_{1,1}$

$$x^{\Delta}(t) = t, \qquad x(0) = x_0$$

$$\Updownarrow$$

$$x(t) = \frac{1}{2}(t^2 - k) + x_0, \quad t \in [2k, 2k+1], \; k \geqslant 0.$$

Example

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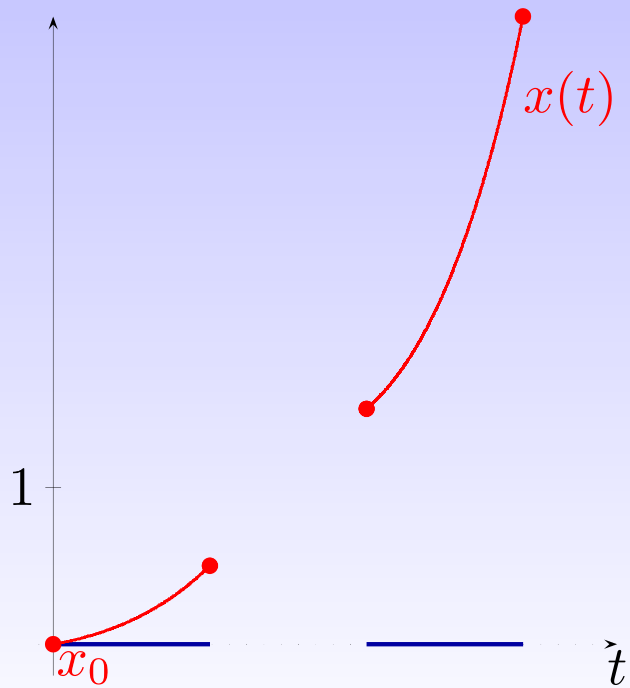
$$\Updownarrow$$

$$x(t) = \frac{1}{2}(t^2 - k) + x_0, \quad t \in [2k, 2k+1], \; k \geqslant 0.$$



Example

$$x_1^{\Delta} = x_2 - x_1$$
$$x_2^{\Delta} = -3 \cdot x_2 - x_1$$

Example

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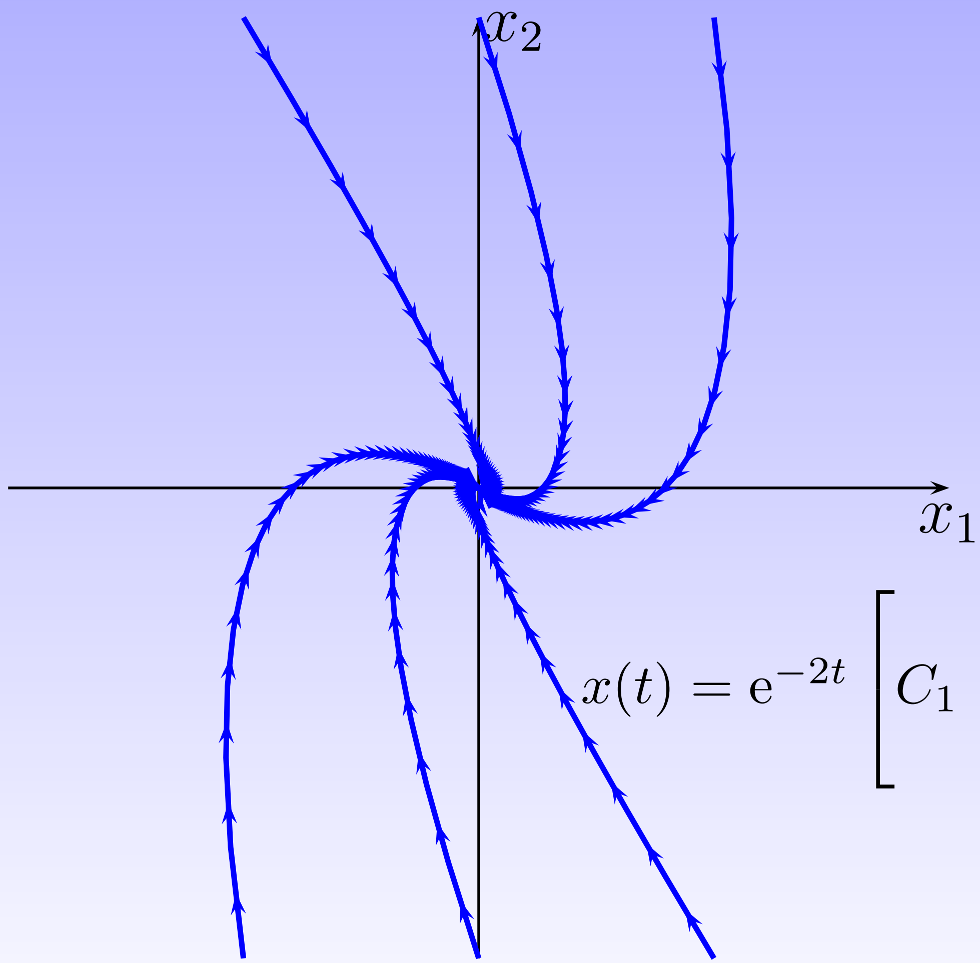
$$\mathbb{T} = \mathbb{R}$$



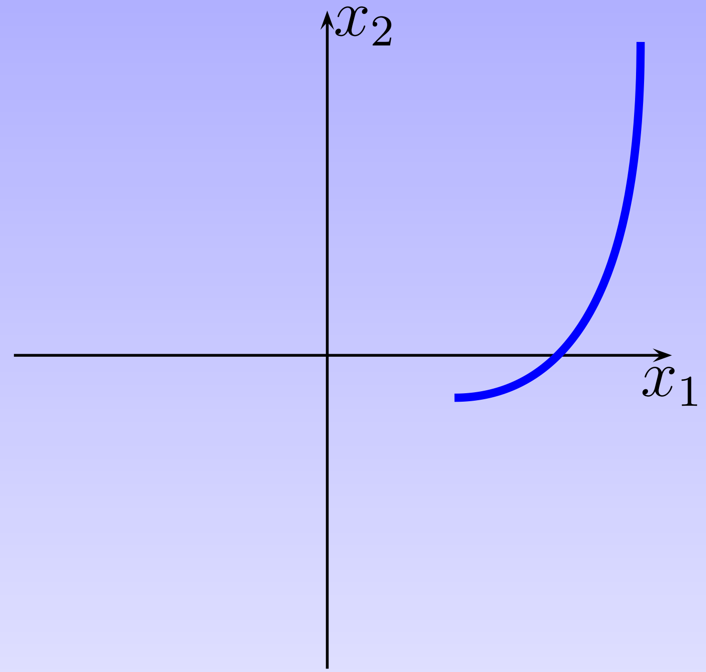
$$x(t) = \mathrm{e}^{-2t}\left[C_1\begin{pmatrix}1+t\\ -t\end{pmatrix} + C_2\begin{pmatrix}t\\ 1-t\end{pmatrix}\right]$$

Example

$$x_1^\Delta = x_2 - x_1$$
$$x_2^\Delta = -3 \cdot x_2 - x_1$$

$$\mathbb{T} = \mathbb{P}_{1,1}$$

$$t \in [0, 1]$$

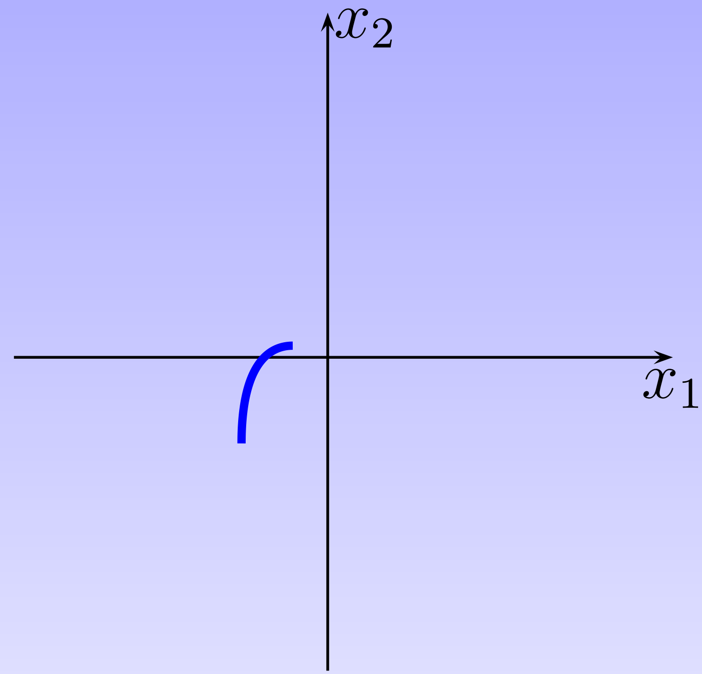


$$x(t) = (-1)^k e^{-2t} \left[C_1 \begin{pmatrix} 1 + t - 2k \\ -t + 2k \end{pmatrix} + C_2 \begin{pmatrix} t - 2k \\ 1 - t + 2k \end{pmatrix} \right]$$

Example

$$x_1^\Delta = x_2 - x_1$$
$$x_2^\Delta = -3 \cdot x_2 - x_1$$

$$\mathbb{T} = \mathbb{P}_{1,1}$$
$$t \in [2, 3]$$



$$x(t) = (-1)^k e^{-2t} \left[C_1 \begin{pmatrix} 1 + t - 2k \\ -t + 2k \end{pmatrix} + C_2 \begin{pmatrix} t - 2k \\ 1 - t + 2k \end{pmatrix} \right]$$

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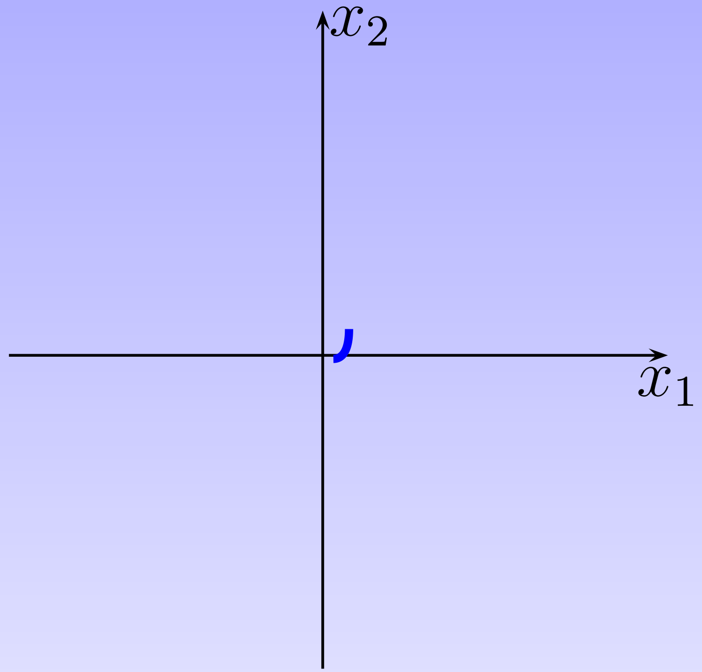
Example

$$\begin{aligned} x_1^\Delta &= x_2 - x_1 \\ x_2^\Delta &= -3 \cdot x_2 - x_1 \end{aligned}$$

$\mathbb{T} = \mathbb{P}_{1,1}$

$t \in [4, 5]$



$$x(t) = (-1)^k e^{-2t} \left[C_1 \begin{pmatrix} 1 + t - 2k \\ -t + 2k \end{pmatrix} + C_2 \begin{pmatrix} t - 2k \\ 1 - t + 2k \end{pmatrix} \right]$$

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Exponential stability

$$x^{\Delta}(t) = Ax(t),$$

$t \in \mathbb{T}$, $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $x(t) \in \mathbb{R}^n$.

$S(\mathbb{T}) := S_{\mathbb{C}}(\mathbb{T}) \cup S_{\mathbb{R}}(\mathbb{T})$ - ***set of exponential stability***

$$S_{\mathbb{C}}(\mathbb{T}) := \{\lambda \in \mathbb{C} \mid \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \to \mu(t)} \frac{\log|1 + s\lambda|}{s} \Delta t < 0\}$$

$$|\cap$$

$$\{\lambda \in \mathbb{C} \mid Re\lambda < 0\}$$

$$S_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} \mid \forall T \in \mathbb{T}\, \exists t \in \mathbb{T},\, t > T : 1 + \mu(t)\lambda = 0\} \subseteq (-\infty, 0)$$

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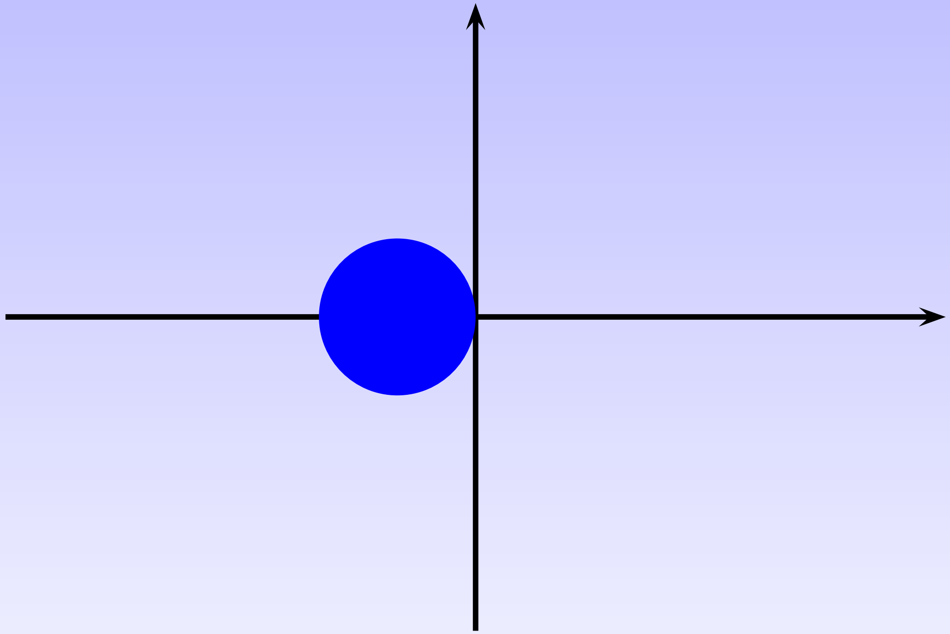
The stability set in discrete- and continuous-time case

$$\frac{x(kh+h)-x(kh)}{h} = Ax(kh), \qquad k \in \mathbb{Z}$$

The stability set:

$h = 2$

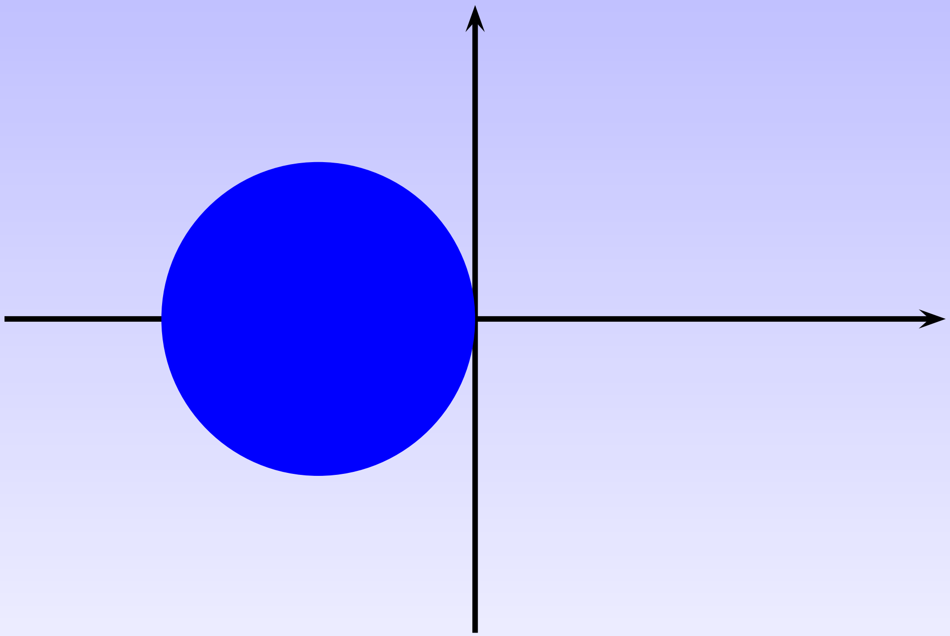


The stability set in discrete- and continuous-time case

$$\frac{x(kh+h)-x(kh)}{h} = Ax(kh), \qquad k \in \mathbb{Z}$$

The stability set:

$h = 1$



The stability set in discrete- and continuous-time case

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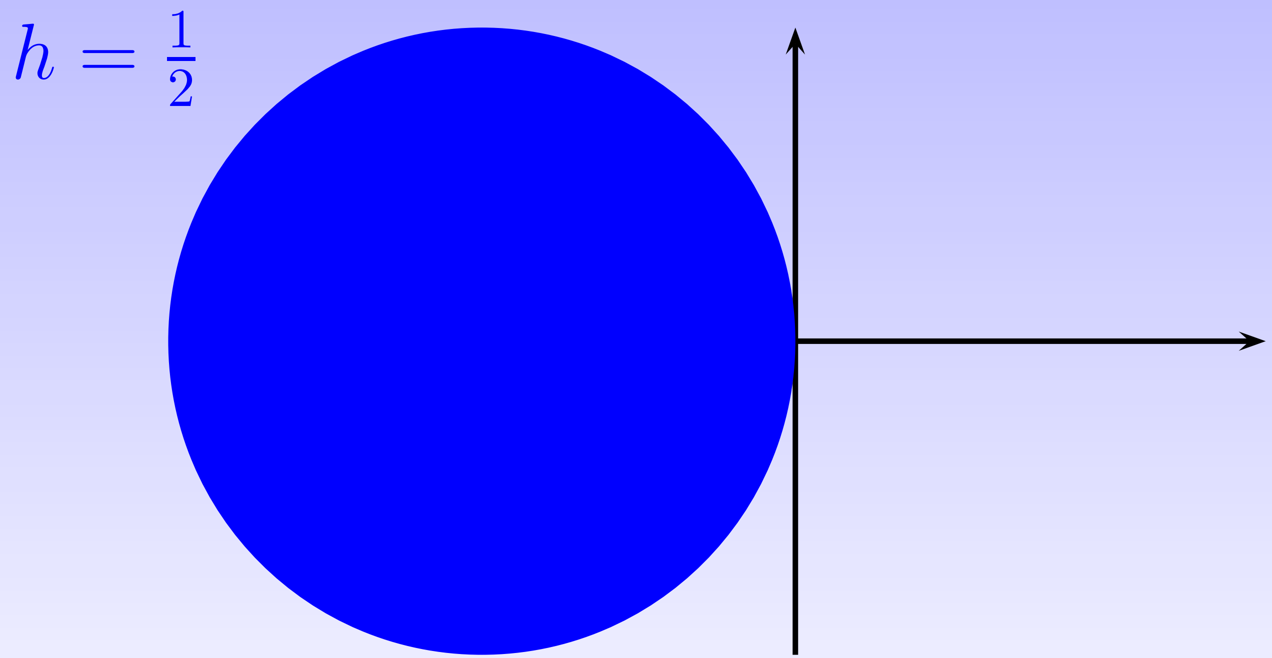
The stability set:

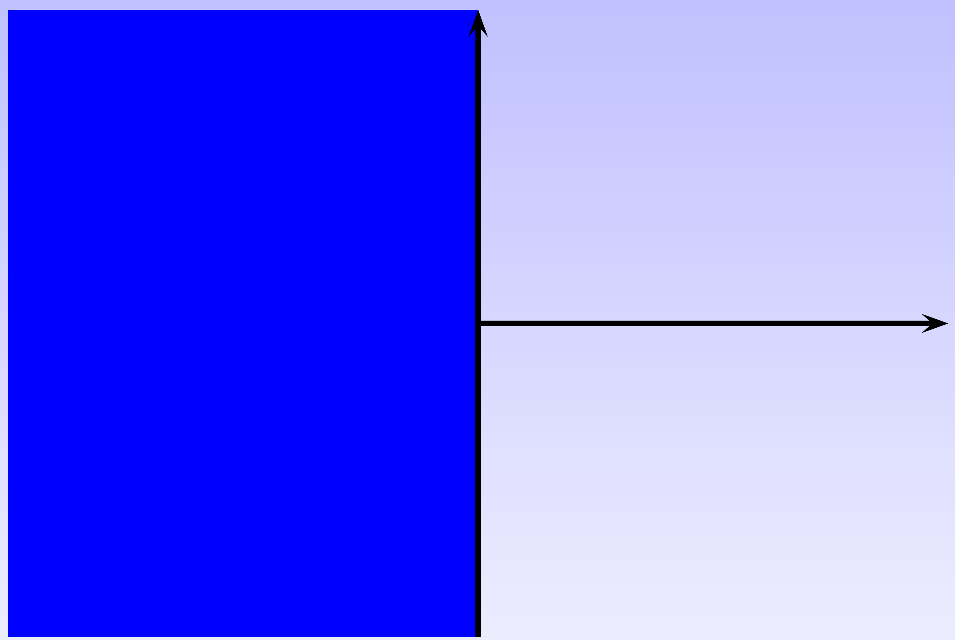


The stability set in discrete- and continuous-time case

$$\frac{x(kh+h)-x(kh)}{h} = Ax(kh), \qquad k \in \mathbb{Z}$$

The stability set:

$h \to 0$



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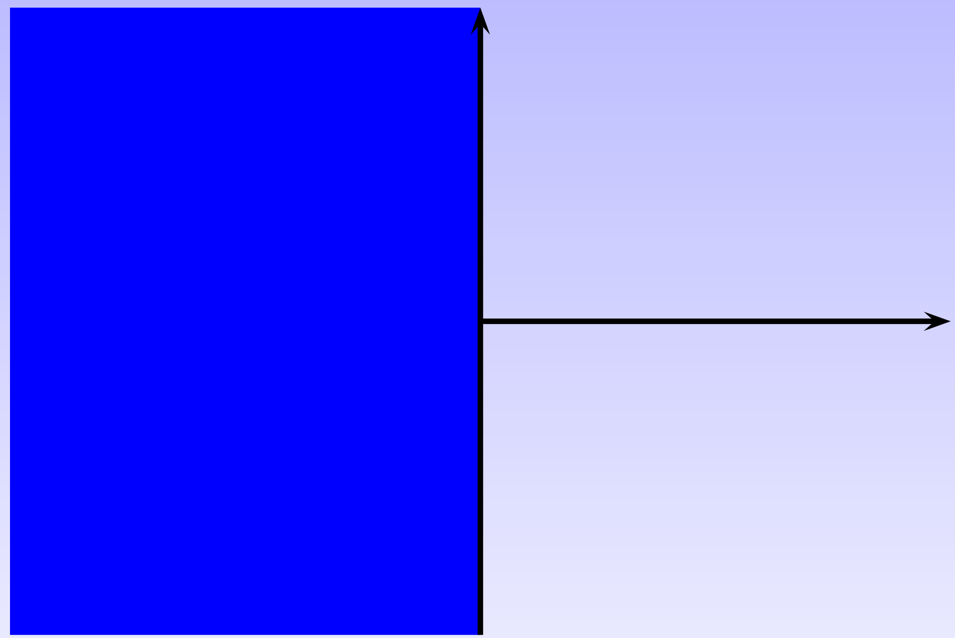
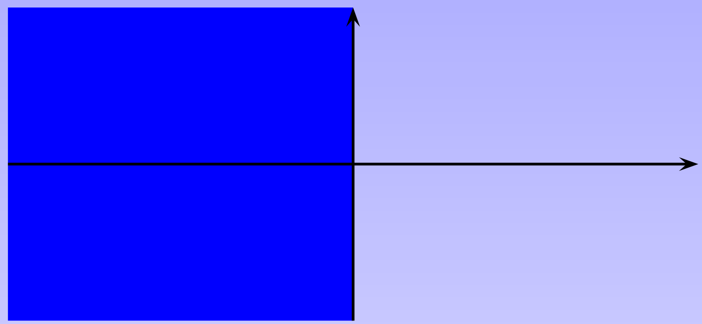
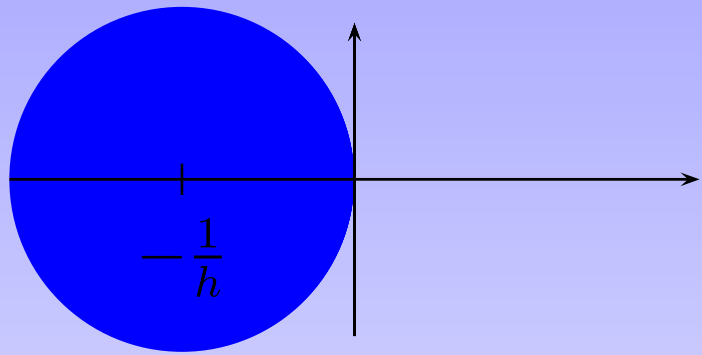
The stability set in discrete- and continuous-time case

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R}$$

The stability set:



Examples

	$\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{H}$	$\mathbb{T} = h\mathbb{Z},\ \ h > 0$
$S(\mathbb{T})$		

☞ $\mathbb{T} = \mathbb{P}_{1,1}$

$$S(\mathbb{T}) = S_{\mathbb{C}}(\mathbb{P}_{1,1}) = \{\lambda \in \mathbb{C} :\ Re\lambda + \log|1+\lambda| < 0\} \cup S_{\mathbb{R}}(\mathbb{P}_{1,1}) = \{-1\}$$

☞ $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \ldots\}, q > 1 \quad \Rightarrow \quad S(\mathbb{T}) = \emptyset$

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Properties of $S(\mathbb{T})$

Let $\sigma_e(A)$ denote the set of all eigenvalues of A.

- ☞ If μ is bounded, then $S_{\mathbb{C}}(\mathbb{T}) \neq \emptyset$.
- ☞ If μ is increasing, then $S_{\mathbb{R}}(\mathbb{T}) = \emptyset$.
- ☞ $x^{\Delta}(t) = Ax(t)$ is exponentially stable $\Rightarrow$ $\sigma_e(A) \subset S(\mathbb{T})$
- ☞ Let A is diagonalizable, then

  $x^{\Delta}(t) = Ax(t)$ is exponentially stable $\Leftrightarrow$ $\sigma_e(A) \subset S(\mathbb{T})$.

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Stabilization and Controllability

$$\Sigma : \boxed{x^{\Delta}(t) = Ax(t) + Bu(t)}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

Σ is ***stabilizable*** if $\exists u(t) = Kx(t)$, $K \in \mathbb{R}^{m\times n}$:
$x^{\Delta}(t) = (A + BK)x(t)$ is exponentially stable.

Σ is ***controllable*** if for any two states $x_1, x_2 \in \mathbb{R}^n$ there exist $t_1, t_2 \in \mathbb{T}$, $t_1 < t_2$, and a piecewise rd-continuous control $u(t)$, $t \in [t_1, t_2] \cap \mathbb{T}$ such that for $x_1 = x(t_1)$ one has $x(t_2) = x_2$.

Stabilization and Controllability

$$\Sigma : \boxed{x^{\Delta}(t) = Ax(t) + Bu(t)}$$

Theorem: Σ is controllable $\Leftrightarrow$

$$\boxed{\operatorname{rank}\left[B, AB, A^2B, \ldots, A^{n-1}B\right] = n.}$$

Assume that μ is bounded. Then

Theorem: Σ is controllable $\Rightarrow$ Σ is stabilizable.

Theorem: Σ is stabilizable $\Leftrightarrow$

$$\boxed{\operatorname{rank}\left[\lambda I - A, B\right] < n \Rightarrow \lambda \in S(\mathbb{T}).}$$

Higher order delta derivatives

$$f^{\overbrace{\Delta \ldots \Delta}^{n-\text{times}}} =: f^{[n]}$$

$f^{[n]}$ is the n-th order delta derivative of f.

Higher order delta derivatives for homogeneous time scales

☞ Let $\mathbb{T} = \mathbb{R}$. Then for a function $f : \mathbb{R} \to \mathbb{R}$ we have

$$f^{(n)} = f^{\overbrace{\Delta \ldots \Delta}^{n-\text{times}}} = f^{[n]}.$$

Higher order delta derivatives for homogeneous time scales

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$$f^{(n)} = f^{\overbrace{\Delta \dots \Delta}^{n-\text{times}}} = f^{[n]}.$$

- Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\sigma^n := \underbrace{\sigma \circ \dots \circ \sigma}_{n-\text{times}}$, $f^{\sigma^n} := f \circ \sigma^n$.

  Then for a function $f : h\mathbb{Z} \to \mathbb{R}$ we have

$$f(t + nh) = f^{\sigma^n}(t) = \sum_{k=0}^{n} \binom{n}{k} h^k f^{[k]}(t).$$

Unification of input-output equations

Continuous-time input-output equation

$$y^{(n)} = \Phi_1(y, \ldots, y^{(n-1)}, u, \ldots, u^{(s)})$$

Discrete-time control input-output equation

$$y(t+n) = \Phi_2(y(t), \ldots, y(t+n-1), u(t), \ldots, u(t+s)) \qquad t \in \mathbb{Z}$$

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$$y(t+n) = \Phi_2(y(t), \dots, y(t+n-1), u(t), \dots, u(t+s)) \qquad t \in \mathbb{Z}$$

Note that for $k, \ell \geqslant 0$ $y^{(k)} = y^{[k]}$, $u^{(\ell)} = u^{[\ell]}$,

$$y(t+k) = \sum_{i=0}^{k} \binom{k}{i} y^{[i]}(t) \text{ and } u(t+\ell) = \sum_{j=0}^{\ell} \binom{\ell}{j} u^{[j]}(t).$$

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$$y^{[n]} = \Phi(y, \ldots, y^{[n-1]}, u, \ldots, u^{[s]})$$

Example of delta derivatives

Let $q > 1$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k \mid k \in \mathbb{Z}\} \cup \{0\}$. Then

$$\sigma(t) = q \cdot t \quad \text{and} \quad \mu(t) = (q-1)t,$$

for all $t \in \mathbb{T}$.
For a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad \text{for all } t \in \mathbb{T} \setminus \{0\}$$

and

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s},$$

provided this limit exists.

The second order delta derivative of f can be computed as follows

$$\begin{aligned} f^{[2]}(t) &= \frac{f^\Delta(\sigma(t))-f^\Delta(t)}{\mu(t)} = \frac{f^\Delta(qt)-f^\Delta(t)}{(q-1)t} = \\ &= \frac{1}{(q-1)t}\left(\frac{f(q^2t)-f(qt)}{(q-1)qt} - \frac{f(qt)-f(t)}{(q-1)t}\right) = \\ &= \frac{f(q^2t)-(1+q)f(qt)+qf(t)}{q(q-1)^2t^2}, \end{aligned}$$

for $t \neq 0$ and

$$f^{[2]}(0) = \lim_{s\to 0}\frac{f^\Delta(s) - f^\Delta(0)}{s} = \lim_{s\to 0}\frac{\frac{f(qs)-f(s)}{(q-1)s} - f^\Delta(0)}{s}$$

provided that both $f^\Delta(0)$ and this limit exist.

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$$f^{[2]}(0) = \lim_{s\to 0} \frac{f^{\Delta}(s) - f^{\Delta}(0)}{s} = \lim_{s\to 0} \frac{\frac{f(qs)-f(s)}{(q-1)s} - f^{\Delta}(0)}{s}$$

provided that both $f^{\Delta}(0)$ and this limit exist.

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Input-output equations

From now we are interested in a higher order delta differential equation on homogeneous time scale $\mathbb{T}$

$$\Sigma : \quad y^{[n]} = \Phi\left(y, y^{\Delta}, \ldots, y^{[n-1]}, u, u^{\Delta}, \ldots, u^{[s]}\right)$$

- ☞ $s < n$,
- ☞ Φ is analytic.

Realization problem

For given

$$\Sigma: \; y^{[n]} = \Phi\left(y, y^{\Delta}, \ldots, y^{[n-1]}, u, u^{\Delta}, \ldots, u^{[s]}\right),$$

find state equations

$$\widetilde{\Sigma}: \quad \begin{aligned} x^{\Delta} &= f(x, u) \\ y &= h(x), \end{aligned}$$

such that $x \in \mathbb{R}^n$ and $\widetilde{\Sigma}$ is ***observable***, i.e.

$$\operatorname{rank} \frac{\partial \left(h(x), h^{\Delta}(x,u), \ldots, h^{[n-1]}(x, u, \ldots, u^{[n-2]})\right)^{\mathrm{T}}}{\partial x} = n.$$

Realization problem

For given

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find state equations

$$\widetilde{\Sigma}: \begin{array}{rcl} x^{\Delta} & = & f(x,u) \\ y & = & h(x), \end{array}$$

such that $x \in \mathbb{R}^{n}$ and $\widetilde{\Sigma}$ is ***observable***, i.e.

$$\operatorname{rank} \frac{\partial \left(h(x), h^{\Delta}(x,u), \ldots, h^{[n-1]}(x,u,\ldots,u^{[n-2]})\right)^{\mathrm{T}}}{\partial x} = n.$$

Field of meromorphic functions

Let $\mathcal{K}$ be the field of meromorphic functions in a finite number of variables from the following set

$$\mathcal{C} = \left\{ y^{[i]}, i = 0, \dots, n-1, \;\; u^{[k]}, \; k \geqslant 0 \right\}.$$

Irreducibility problem

$\varphi_r \in \mathcal{K}$ is said to be ***autonomous element*** for

$$\Sigma: \; y^{[n]} = \phi\left(y, y^{[1]}, \dots, y^{[n-1]}, y, u^{[1]}, \dots, u^{[s]}\right),$$

if $\exists \; \nu \geqslant 1$ and a non-zero analytic function F:

$$F\left(\varphi_r, \varphi_r^{\Delta}, \dots, \varphi_r^{[\nu]}\right) = 0.$$

Σ is called ***irreducible*** if there does not exist any non-zero autonomous element for it in $\mathcal{K}$.

Otherwise Σ is ***reducible***.

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Operator $\sigma : \mathcal{K} \to \mathcal{K}$

Let $\sigma : \mathcal{K} \to \mathcal{K}$ be an operator defined by

$$\sigma(F)\left(y, \ldots, y^{[n-1]}, u, \ldots, u^{[\ell+1]}\right) :=$$

$$:= F\left(y^{\sigma}, \ldots, \left(y^{[n-1]}\right)^{\sigma}, u^{\sigma}, \ldots, \left(u^{[\ell]}\right)^{\sigma}\right),$$

where

$$y^{\sigma} = y + \mu y^{\Delta}, \quad \left(y^{[n-1]}\right)^{\sigma} = y^{[n-1]} + \mu\Phi\left(y^{[0..n-1]}, u^{[0..s]}\right),$$

$$u^{\sigma} = u + \mu u^{\Delta}, \quad \left(u^{[\ell]}\right)^{\sigma} = u^{[\ell]} + \mu u^{[\ell+1]}.$$

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where

$$y^\sigma = \boxed{y + \mu y^\Delta}, \quad \left(y^{[n-1]}\right)^\sigma = \boxed{y^{[n-1]} + \mu\Phi\left(y^{[0..n-1]}, u^{[0..s]}\right)},$$

$$u^\sigma = u + \mu u^\Delta, \quad \left(u^{[\ell]}\right)^\sigma = u^{[\ell]} + \mu u^{[\ell+1]}.$$

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$$:= F\left(y^\sigma, \ldots, \left(y^{[n-1]}\right)^\sigma, \underline{u^\sigma}, \ldots, \underline{\left(u^{[\ell]}\right)^\sigma}\right),$$

where

$$y^\sigma = y + \mu y^\Delta, \quad \left(y^{[n-1]}\right)^\sigma = y^{[n-1]} + \mu\Phi\left(y^{[0..n-1]}, u^{[0..s]}\right),$$

$$u^\sigma = \boxed{u + \mu u^\Delta}, \quad \left(u^{[\ell]}\right)^\sigma = \boxed{u^{[\ell]} + \mu u^{[\ell+1]}}.$$

Operator $\Delta : \mathcal{K} \to \mathcal{K}$

Operator Δ satisfies a generalization of Leibniz rule, i.e.

$$(FG)^{\Delta} = F^{\sigma} G^{\Delta} + F^{\Delta} G. \qquad (*)$$

A derivation Δ satisfying the rule $(*)$ is called a ***"σ-derivation"***.

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A derivation Δ satisfying the rule $(*)$ is called a "***σ-derivation***".

Space of one-forms

Define

$$\mathcal{E} := \operatorname{span}_{\mathcal{K}} \left\{ \mathrm{d}y^{[i]},\ i = 0, \ldots, n-1,\ \ \mathrm{d}u^{[\ell]},\ \ \ell \geqslant 0 \right\}.$$

The elements of $\mathcal{E}$ are called ***one-forms***.

If $\omega \in \mathcal{E}$, then

$$\omega = \sum_{i=0}^{n-1} A_i \mathrm{d}y^{[i]} + \sum_{\ell \geqslant 0} B_\ell \mathrm{d}u^{[\ell]}.$$

Operators Δ and σ on $\mathcal{E}$

The operators $\Delta : \mathcal{K} \to \mathcal{K}$ and $\sigma : \mathcal{K} \to \mathcal{K}$ induce operators $\Delta : \mathcal{E} \to \mathcal{E}$ and $\sigma : \mathcal{E} \to \mathcal{E}$ by

$$\Delta\left(\sum_i \alpha_i \mathrm{d}\zeta_i\right) := \sum_i \left\{\alpha_i^{\Delta} \mathrm{d}\zeta_i + \alpha_i^{\sigma} \mathrm{d}\left[\zeta_i^{\Delta}\right]\right\},$$

$$\sigma\left(\sum_i \alpha_i \mathrm{d}\zeta_i\right) := \sum_i \alpha_i^{\sigma} \mathrm{d}\left[\zeta_i^{\sigma}\right],$$

where

$$\alpha_i \in \mathcal{K} \text{ and } \zeta_i \in \mathcal{C} = \{y, \ldots, y^{[n-1]}, u^{[k]} : k \geqslant 0\}.$$

The subspaces of one-forms

Introduce the sequence of subspaces

$$\mathcal{H}_1 \supset \cdots \supset \mathcal{H}_{k-1} \supset \mathcal{H}_k = \mathcal{H}_{k+1} = \cdots =: \mathcal{H}_\infty \,.$$

of $\mathcal{E}$ defined by

$$\mathcal{H}_1 = \operatorname{span}_{\mathcal{K}} \left\{ \mathrm{d}y^{[i]}, \mathrm{d}u^{[j]},\ i = 0, \ldots, n-1,\ j = 0, \ldots, s \right\},$$

$$\mathcal{H}_k = \operatorname{span}_{\mathcal{K}} \left\{ \omega \in \mathcal{H}_{k-1} \mid \omega^\Delta \in \mathcal{H}_{k-1} \right\}, \quad k \geqslant 2.$$

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Realizabilitity condition

Theorem: The nonlinear system Σ described by i/o delta differential equation on homogeneous time scales $\mathbb{T}$

$$\Sigma: \; y^{[n]} = \Phi\left(y, y^{\Delta}, \ldots, y^{[n-1]}, u, u^{\Delta}, \ldots, u^{[s]}\right),$$

has an observable state-space realization of the form

$$\begin{aligned} \widetilde{\Sigma}: \; x^{\Delta} &= f(x,u) \\ y &= h(x), \end{aligned}$$

if and only if the subspaces $\mathcal{H}_k$ are completely integrable, for $1 \leqslant k \leqslant s+2$.

Irreducibility condition

Theorem:

$$\Sigma: \; y^{[n]} = \Phi\left(y, y^{\Delta}, \ldots, y^{[n-1]}, u, u^{\Delta}, \ldots, u^{[s]}\right),$$

is irreducible if and only if $\mathcal{H}_\infty = \{0\}$.

Example

Consider the i/o delta-differential equation on homogeneous time scale $\mathbb{T}$

$$\Sigma: \; y^{[2]} = \frac{u^{\Delta} y^{\Delta}}{u} + u y^2 + \mu u^{\Delta} y^2.$$

Then

$$\mathcal{H}_1 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u, \mathrm{d}u^{\Delta}\}$$

and

$$\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u\}.$$

Since $\mathrm{d}y^{\Delta} \in \mathcal{H}_2$,

$$\mathrm{d}y \in \mathcal{H}_3.$$

However, note that $\mathrm{d}u^{\Delta} \notin \mathcal{H}_2$ and

$$\mathrm{d}y^{[2]} = (2uy + 2\mu u^{\Delta} y)\,\mathrm{d}y + \left(y^2 - \frac{u^{\Delta} y^{\Delta}}{u^2}\right)\mathrm{d}u + \\ + \frac{u^{\Delta}}{u}\mathrm{d}y^{\Delta} + \left(\frac{y^{\Delta}}{u} + \mu y^2\right)\mathrm{d}u^{\Delta} \notin \mathcal{H}_2,$$

so

$$\mathrm{d}u, \mathrm{d}y^{\Delta} \notin \mathcal{H}_3.$$

One can check that $u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u \in \mathcal{H}_3$.

Therefore

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u\}$$

and $\mathcal{H}_3$ is integrable.

One-form $u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u$ can be made exact by multiplying with integrating factor $\frac{1}{u^2}$.

Then

$$\frac{1}{u}\mathrm{d}y^{\Delta} - \frac{y^{\Delta}}{u^2}\mathrm{d}u = \mathrm{d}\left[\frac{y^{\Delta}}{u}\right].$$

The state coordinates can be chosen as follows

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{y^{\Delta}}{u} \end{aligned}$$

In these coordinates the system has the classical state space form

$$\widetilde{\Sigma}: \begin{aligned} x_1^{\Delta} &= x_2 u \\ x_2^{\Delta} &= x_1^2 \\ y &= x_1. \end{aligned}$$

Moreover

$$\left(u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u\right)^{\Delta} =$$

$$= \frac{u + u^{\sigma}}{u^2} u^{\Delta} \left(u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u\right) + 2y\left(u^{\sigma}\right)^2 \mathrm{d}y \in \mathcal{H}_3.$$

Hence

$$\mathcal{H}_4 = \mathrm{span}_{\mathcal{K}} \left\{u\mathrm{d}y^{\Delta} - y^{\Delta}\mathrm{d}u\right\}$$

and

$$\mathcal{H}_5 = \{0\}.$$

Moreover

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The considered system Σ is irreducible.

Polynomial description of system Σ

$$y^{[n]} = \phi\left(y, \ldots, y^{[n-1]}, u, \ldots, u^{[s]}\right)$$

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$\downarrow$ d

$$\mathrm{d}y^{[n]} - \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \mathrm{d}y^{[i]} - \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \mathrm{d}u^{[j]} = 0.$$

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$\partial^k \mathrm{d}y := \mathrm{d}y^{[k]}$ $\partial^l \mathrm{d}u := \mathrm{d}u^{[l]}$

$$\left(\partial^n - \sum_{i=1}^{n-1} \frac{\partial\phi}{\partial y^{[i]}} \partial^i\right) \mathrm{d}y = \left(\sum_{j=0}^{s} \frac{\partial\phi}{\partial u^{[j]}} \partial^j\right) \mathrm{d}u,$$

where $\dfrac{\partial\phi}{\partial y^{[i]}} \in \mathcal{K}, \quad \dfrac{\partial\phi}{\partial u^{[j]}} \in \mathcal{K}.$

Let $p(\partial) := \partial^n - \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \partial^i$ and $q(\partial) := \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \partial^j$.

Then

$$\Sigma : \quad p(\partial)\mathrm{d}y = q(\partial)\mathrm{d}u.$$

Since $\frac{\partial \phi}{\partial y^{[i]}}, \frac{\partial \phi}{\partial u^{[j]}} \in \mathcal{K}$, $p(\partial)$ and $q(\partial)$ are elements of the ring of left differential polynomials $\mathcal{K}[\partial; \sigma, \Delta]$, where

$$\partial \cdot A := A^\sigma \cdot \partial + A^\Delta,$$

for $A \in \mathcal{K}$.

Irreducibility condition

Theorem: Control system

$$y^{[n]} = \phi\left(y, y^{\Delta}, \ldots, y^{[n-1]}, u, u^{\Delta}, \ldots, u^{[s]}\right)$$

is irreducible if and only if polynomials

$$p(\partial) = \partial^n - \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \partial^i \quad \text{and} \quad q(\partial) = \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \partial^j$$

are relatively left prime.

Example 1

$$\Sigma : \quad y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + y^{\Delta} - uy = 0$$

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Applying the Euclidean division algorithm, we get

$$\overbrace{\partial^2 + (1 - u^{\sigma})\partial - u^{\Delta} - u} = \overbrace{(y^{\sigma}\partial + y^{\Delta} + y)}\left(\frac{1}{y}\partial - \frac{u}{y}\right).$$

So $q(\partial) = y^\sigma \partial + y^\Delta + y$ is the greatest common left divisor of $p(\partial) = \partial^2 + (1 - u^\sigma)\partial - u^\Delta - u$ and $q(\partial)$.

Then

$$\left(\frac{1}{y}\partial - \frac{u}{y}\right) \mathrm{d}y = \mathrm{d}u$$

or alternatively $\mathrm{d}\left[y^\Delta - yu\right] = 0.$

The autonomous element for Σ is the function

$$\varphi_r(y, y^\Delta, u) = y^\Delta - yu \quad \text{and} \quad \varphi_r^\Delta + \varphi_r = 0.$$

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The autonomous element for Σ is the function

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The considered system Σ is reducible.

Example 2

$$\Sigma : \quad y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + yy^{\Delta} - uy^2 = 0$$

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$$\Sigma : \; y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + yy^{\Delta} - uy^2 = 0$$

d

$$\left(\partial^2 + (y - u^{\sigma})\partial \; - u^{\Delta} + y^{\Delta} - 2uy\right) \mathrm{d}y = \\ = \left(y^{\sigma}\partial + y^{\Delta} + y^2\right) \mathrm{d}u$$

Applying the Euclidean division algorithm, we get

$$\partial^2 + (y - u^\sigma)\partial - u^\Delta + y^\Delta - 2uy =$$
$$= \left(y^\sigma \partial + y^\Delta + y^2\right)\left(\frac{1}{y}\partial - \frac{u}{y}\right) + y^\Delta - uy.$$

Therefore polynomials

$$p(\partial) = \partial^2 + (y - u^\sigma)\partial - u^\Delta + y^\Delta - 2uy,\ q(\partial) = y^\sigma \partial + y^\Delta + y^2$$

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are relatively left prime.

The considered system Σ is irreducible.

Conclusions

- Time scale seems to be a perfect language to unify the continuous and the discrete.
- The time scale calculus and some control problems.
- Application of time scales in another branch of the science.
- http://web.ecs.baylor.edu/faculty/marks/Research/TimeScales/index.htm

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Thank you very much